

Optimization Problems in Supply Chain Management

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Optimaliseringsproblemen in Supply Chain Management

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Part I

Optimization
in
Supply Chains

Chapter 1

Introduction

1.1 Distribution in a dynamic environment

Supply Chain Management is a field of growing interest for both companies and researchers. It consists of the management of material, information, and financial flows in a logistics distribution network composed of parties like vendors, manufacturers, distributors, and customers. The environment in which companies nowadays manage their supply chain is highly dynamic. This thesis is devoted to the development of optimization tools that enable companies to detect, and subsequently take advantage of, opportunities that may exist for improving the efficiency of their logistics distribution networks in such a dynamic environment.

Within the field of distribution logistics a number of developments have occurred over the past years. We have seen a globalization of supply chains in which national boundaries are becoming less important. Within Europe we can observe an increase in the attention that is being paid by West-European companies to markets in Eastern Europe and the former Soviet Union. The fact that European borders are disappearing within the European Union results in questions about the reallocation, often concentration, of production. Moreover, the relevance of European and regional distribution centers instead of national ones is reconsidered. According to Kotabe [80], the national boundaries are losing their significance as a psychological and physical barrier to international business. Therefore, companies are stimulated to expand their supply chains across different countries. Global supply chains try to take advantage of the differences in characteristics of various countries when designing their manufacturing and sourcing strategies. For example, the labor and raw materials costs are lower in developing countries while the latest advances in technology are present only in developed countries. As pointed out by Vidal and Goetschalckx [135], global supply chains are more complex than domestic ones because, in an international setting, the flows in the supply chain are more difficult to coordinate. Issues that are exclusive to a global supply chain are different taxes, trade barriers, and transfer prices.

There are dynamics inherent to the flows in the supply chain. Fisher [46] in-

roduces the concepts *functional* and *innovative* to classify products. Functional products are the physical products without any added value in the form of, for instance, special packaging, fashionable design, service, etc. He argues that functional products have a relatively long life cycle, and thus a stable and steady demand, but often also low profit margins. Therefore, companies introduce innovations in fashion or technology with the objective of creating a competitive advantage over other suppliers of physically similar products, thereby increasing their margins. As a consequence, this leads to a shortening of the life cycle of innovative products since companies are forced to introduce new innovations to stay competitive. Another development that can be observed is the customer orientation. Supply chains have to satisfy the requirements of the customers with respect to the customized products as well as the corresponding services. The first step when designing and controlling an effective supply chain is to investigate the nature of the demand of the products. The tendency towards a shorter life cycle for innovative products leads to highly dynamic demand patterns and companies need to regularly reconsider the design of their supply chains to effectively utilize all the opportunities for profit.

One way of creating a competitive advantage is by maintaining a highly effective logistics distribution network. Thus, logistics becomes an integral part of the product that is being delivered to the customer. Competitiveness encourages a continuous improvement of the customer service level, see Ballou [6]. For example, one of the most influencing elements in the quality of the customer service is the lead time. Technological advances can be utilized to reduce these lead times, see Slats et al. [125]. For example, Electronic Data Interchange with or without internet leads to improved information flows, which yield a better knowledge of the customers' needs at each stage of the supply chain.

Another way that a company can add value to its products to distinguish them from competing products is to take environmental issues into account, thereby creating a so-called 'green supply chain', see Thomas and Griffin [130]. For further references on that topic see e.g. Bloemhof-Ruwaard et al. [18] and Fleischmann et al. [50].

The outline of this chapter is as follows. In Section 1.2 we describe the opportunities for increasing the efficiency of the supply chain that can be achieved by using Operations Research techniques. In Section 1.3 we will discuss the importance of coordination in the supply chain. In Section 1.4 we will state the necessity of a dynamic model when combining transportation and inventory decisions, or when dealing with products that show a strong seasonal component in their demand patterns or suffer from perishability. Finally, in Section 1.5 we will describe the goal of the thesis and we will briefly describe its contents. Some of the discussions in this chapter can be found in Romero Morales, Van Nunen and Romeijn [117].

1.2 Operations Research in Supply Chain Management

The mission of Operations Research is to support real-world decision-making using mathematical and computer modeling, see Luss and Rosenwein [86]. Supply Chain Management is one of the areas where Operations Research has proved to be a powerful tool, see Geunes and Chang [60] and Tayur, Ganeshan, and Magazine [129].

Bramel and Simchi-Levi [20] claim that, in logistics management practice, the tendency to use decision rules that were adequate in the past, or that seem to be intuitively good, is still often observed. However, it proved to be worthwhile using scientific approaches to certificate a good performance of the supply chain or to detect opportunities for improving it. Many times this leads to a more effective performance of the supply chain while maintaining or even improving the customer service level. There are many examples of different scientific approaches used in the development of decision support systems (see e.g. Van Nunen and Benders [134], Benders et al. [13], and Hagdorn-van der Meijden [67]), or the development of new optimization models representing the situation at hand as closely as possible (see e.g. Geoffrion and Graves [58], Gelders, Pintelon and Van Wassenhove [56], Fleischmann [49], Chan, Muriel and Simchi-Levi [29], Klose and Stähly [77], and Tüshaus and Wittmann [132]).

Geoffrion and Powers [59] summarize some of the main reasons for the increasing role of optimization techniques in the design of distribution systems. The most crucial one is the development of the capabilities of computers that allow for the investigation of richer and more realistic models than could be analyzed before. In these extended models, additional important issues, for e.g. scenario analysis, can be included. This development in computer technology is accompanied by new advances in algorithms, see Nemhauser [99].

The vast literature devoted to quantitative methods in Supply Chain Management also suggests the importance of Operations Research in this field. Bramel and Simchi-Levi [20] have shown the power of probabilistic analysis when defining heuristic procedures for distribution models. Geunes and Chang [60] give a survey of models in Operations Research emphasizing the design of the supply chain and the coordination of decisions. Tayur, Ganeshan and Magazine [129] have edited a book on quantitative models in Supply Chain Management. The last chapter of this book is devoted to a taxonomic review of the Supply Chain Management literature.

In a more practical setting, Gelders, Pintelon and Van Wassenhove [56] use a plant location model for the reorganization of the logistics distribution networks of two small breweries into a single bigger one. Shapiro, Singhal and Wagner [122] develop a Decision Support System based on Mathematical Programming tools to consolidate the value chains of two companies after the acquisition of the second by the first one. Arntzen et al. [4] present a multi-echelon multi-period model which was used in the reorganization of Digital Equipment Corporation. Hagdorn-van der Meijden [67] presents some examples of companies where new structures have been implemented recently. Köksalan and Süral [79] use a multi-period Mixed Integer Problem for the opening of two new malt plants for Efes Beverage. Myers [95]

presents an optimization model to forecast the demand that a company producing plastic closures can accommodate when these closures suffer from marketing perishability. From an environmental point of view, decreasing the freight traffic is highly desirable. Kraus [81] claims that most of the environmental parameters for evaluating transportation in logistics distribution networks are proportional to the total distance traveled, thus a lot of effort is put into developing systems that decrease that distance.

1.3 Coordination in supply chains

A company delivers its products to its customers by using a logistics distribution network. Such a network typically consists of product flows from the producers to the customers through transshipment points, distribution centers (warehouses), and retailers. In addition, it involves a methodology for handling the products in each of the levels of the logistics distribution network, for example, the choice of an inventory policy, or the transportation modes to be used.

Designing and controlling a logistics distribution network involve different levels of decision-making, which are not independent of each other but exhibit interactions. At the operational level, day-to-day decisions like the assignment of the products ordered by individual customers to trucks, and the routing of those trucks must be taken. The options and corresponding costs that are experienced at that level clearly depend on choices that have been made at the longer term tactical level. The time horizon for these tactical decisions is usually around one year. Examples of decisions that have to be made at this level are the allocation of customers to warehouses and how the warehouses are supplied by the plants, the inventory policy to be used, the delivery frequencies to customers, and the composition of the transportation fleet. Clearly, issues that play a role at the operational level can dictate certain choices and prohibit others at the tactical level. For instance, the choice of a transportation mode may require detailed information about the current transportation costs which depend on decisions at the operational level. Similarly, the options and corresponding costs that are experienced at the tactical level clearly depend on the long-term strategic choices regarding the design of the logistics distribution network that have been made. The time horizon for these strategic decisions is often around three to five years. The most significant decisions to be made at this level are the number, the location and the size of the production facilities (plants) and distribution centers (warehouses). But again, issues that play a role at the tactical level could influence the options that are available at the strategic level. When designing the layout of the logistics distribution network we may need detailed information about the actual transportation costs, which is an operational issue as mentioned above.

To ensure an efficient performance of the supply chain, decisions having a significant impact on each other must be coordinated. For instance, companies believe that capacity is expensive (see Bradley and Arntzen [19]). This has a twofold consequence. Firstly, the purchase of production equipment is made by top managers, while the production schedules and the inventory levels are decided at lower levels in the company. Therefore, the coordination between those decisions is often present

to a limit extent. Secondly, expensive equipment is often used to full capacity, which leads to larger inventories than necessary to meet the demand and causes an imbalance between capacity and inventory investments. Bradley and Arntzen [19] propose a model where the capacity and the inventory investments, and the production schedule are integrated and show the opportunities for improvement in the performance of the supply chain found in two companies.

Coordination is not only necessary between the levels of decision-making but also between the different stages of the supply chain, like procurement, production and distribution (see Thomas and Griffin [130]). In the past, these stages were managed independently, buffered by large inventories. The decisions in different stages were often decoupled since taking decisions in just one of these stages was already a comprehensive task in itself. For example, from a computational point of view, just the daily delivery of the demand of a set of customers is a hard problem. Decoupling decisions in different stages causes larger costs and longer delivery times. Nowadays, the fierce competition in the market forces companies to be more efficient by taking decisions in an integrated manner. This has been possible due to the tremendous development of computer capabilities and the new advances in algorithms, see Geoffrion and Powers [59].

Several examples can be found in the literature proving that models coordinating at least two stages of the supply chain can detect new opportunities that may exist for improving the efficiency of the supply chain. For instance, Chandra and Fisher [30] propose two solution approaches to investigate the impact of coordination of production and distribution planning. They consider a single-plant, multi-product, and multi-period scenario. The plant produces and stores products for a while. After that, they are delivered to the retailers (customers) by a fleet of trucks. One of the solution approaches tackles the production scheduling and the routing problems separately. They compare this approach to a coordinated approach where both decisions are incorporated in one model. In their computational study, they show that the coordinated approach can yield up to 20% in costs savings. Anily and Federgruen [3] study a model integrating inventory control and transportation planning decisions motivated by the trade-off between the size and the frequency of delivery. They consider a single-warehouse and multi-retailer scenario where inventory can only be kept at the retailers which face constant demand. The model determines the replenishment policy at the warehouse and the distribution schedule for each retailer so that the total inventory and distribution costs are minimized. They present heuristic procedures to find upper and lower bounds on the optimal solution value.

Models coordinating different stages of the supply chain can be again classified as strategic, tactical or operational. Several surveys can be found in the literature addressing coordination issues. Beamon [11] summarizes models in the area of multi-stage supply chain design and analysis. Erengüç, Simpson and Vakharia [39] survey models integrating production and distribution planning. Thomas and Griffin [130] survey coordination models on strategic and operational planning. Vidal and Goetschalckx [135] pay attention to strategic production-distribution models with special emphasis on global supply chain models. Bhatnagar, Chandra and Goyal [15] call the coordination between the three stages of the supply chain ‘general

coordination'. They also describe the concept of multi-plant coordination, i.e., the coordination of production planning among multiple plants, and survey the literature on this topic.

1.4 A dynamic model for evaluating logistics network designs

When opportunities for improvement in the design of a logistics distribution network are present, management can define alternatives to the current design. In order to be able to evaluate and compare these alternatives, various performance criteria (under various operating strategies) need to be defined. An example of such a criterion could be total operational costs, see Beamon [11].

There are many examples of products where the production and distribution environment is dynamic, for instance because the demand contains a strong seasonal component. For example, the demand for soft drinks and beers is heavily influenced by the weather, leading to a much higher demand in warmer periods. A rough representation of the demand pattern, disregarding the stochastic component due to daily unpredictable/unforeseeable changes in the weather, will show a peak in summer and a valley in winter. Nevertheless, most of the existing models in the literature are static (single-period) in nature. This means that the implicit assumption is made that the environment, including all problem data, are constant over time. For instance, demand patterns are assumed to be constant, even though this is often an unrealistic assumption. Hence, the adequacy of those models is limited to situations where the demand pattern exhibits no remarkable changes throughout the planning horizon. In practice, it means that all, by nature dynamic, input parameters to the model are approximated by static ones like average parameters.

The simplest approach to deal with this issue could be a two-phase procedure where we first solve a suitable static model and in the second phase try to obtain a solution to the actual problem. For example, Van Nunen and Benders [134] use only the data corresponding to the peak season for their medium and long-term analysis. At the other end of the spectrum, an approach could be to define the parameters of the model and the decisions to be taken as functions of time, and impose that capacity constraints must be satisfied at each point in time. The difficulties present in the data estimation, the analysis of the model, as well as the implementation of the solution (if one can be found) make this approach rather impractical. As mentioned above, the literature focuses mainly on static models. There are notable exceptions where the planning horizon is discretized. Duran [37] plans the production, bottling, and distribution to agencies of different types of beer, with an emphasis on the production process. A one year planning horizon is considered, but in contrast to most of the literature, the model is dynamic with twelve monthly periods. Chan, Muriel and Simchi-Levi [29] study a dynamic, but uncapacitated, distribution problem in an operational setting. Arntzen et al. [4] present a multi-echelon multi-period model with no single-sourcing constraints on the assignment variables which was used in the reorganization of Digital Equipment Corporation.

Following those models, we will also discretize the planning horizon, thereby closely approximating the true time-dependent behaviour of the data. We propose splitting the planning horizon into smaller periods where demands in each period are forecasted by constant values. (Note that it is not required that the periods are of equal length!) This means that, implicitly, we are assuming that the demand has a stationary behavior *in each period*.

Another advantage of a dynamic approach to the problem is the ability to explicitly model inventory decisions. This enables us to jointly estimate transportation and inventory costs. Recall that this is not possible when considering an aggregate single-period representation of the problem. Our model can also deal with products having a limited shelf-life, in other words, products that suffer from perishability. This can be caused by the fact that the product exhibits a physical perishability or it might be affected by an economic perishability (obsolescence). In both cases, the storage duration of the product should be limited. Perishability constraints have mainly been taken into account in inventory control, but they can hardly be found in the literature on Physical Distribution. A notable exception is Myers [95], who presents a model where the maximal demand that can be satisfied for a given set of capacities and under perishability constraints is calculated.

1.5 Goal and summary of the thesis

The goal of this thesis is the study of optimization models, which integrate both transportation and inventory decisions, to search for opportunities for improving the logistics distribution network. In contrast to Anily and Federgruen [3], who also study a model integrating these two aspects in a tactical-operational setting, we utilize the models to answer strategic and tactical questions. We evaluate an estimate of the total costs of a given design of the logistics distribution network, including production, handling, inventory holding, and transportation costs. The models are also suitable for clustering customers with respect to the warehouses, and through this they can be used as a first step towards estimating operational costs in the logistics distribution network related to the daily delivery of the customers in tours.

The main focus of this thesis is the search for solution procedures for these optimization models. Their computational complexity makes the use of heuristics solution procedures for large-size problem instances advisable. We will look for feasible solutions with a class of greedy heuristics. For small or medium-size problem instances, we will make use of a Branch and Price scheme. Relevant characteristics of the performance of a solution procedure are the computation time required and the quality of the solution obtained if an optimal solution is not guaranteed. Conclusions about these characteristics are drawn by testing it on a collection of problem instances. The validity of the derived conclusions strongly depends on the set of problem instances chosen for this purpose. Therefore, the second focus of this thesis is the generation of experimental data for these optimization models to test these solution methods adequately.

The well-known generalized assignment problem (GAP) can be seen as a static

model to evaluate a two-level logistics distribution network where production and storage take place at the same location. Moreover, some of the dynamic models analyzed in this thesis can be reformulated as a GAP with a nonlinear objective function. Therefore, we will devote Part II of the thesis to studying this problem. In Parts III and IV we will study variants of a multi-period single-sourcing problem (MPSSP) that can be used for evaluating logistics distribution network designs with respect to costs in a dynamic environment. In particular, Part III is dedicated to the analysis of a logistics distribution network where production and storage take place at the same location and only the production capacity is constrained. In Chapter 10, we will separately add different types of capacity constraints to this model. In particular, we will analyze the addition of constraints with respect to the throughput of products and the volume of inventory. Furthermore, we will analyze how to deal with perishable products. In Chapter 11, we will study models in which the production and storage levels are separated.

In Parts III and IV, the customers' demand patterns for a single product are assumed to be known and each customer needs to be delivered by a unique warehouse in each period. The decisions that need to be made are (i) the production sites and quantities, (ii) the assignment of customers to facilities, and (iii) the location and size of inventories. These decisions can be handled in a nested fashion, where we essentially decide on the assignment of customers to facilities only, and where the production sites and quantities, and the location and size of inventories are determined optimally as a function of the customer assignments. Viewed in this way, the MPSSP is a generalization of the GAP with a convex objective function, multiple resource requirements, and possibly additional constraints, representing, for example, throughput or physical inventory capacities, or perishability of the product.

To be able to deal with many variants of the MPSSP using a single solution approach, we will introduce in Chapter 2 a general class of *convex capacitated assignment problems* (CCAP's). A distinguished member of this class is the GAP. As mentioned above, we are concerned with adequately testing solution procedures for the CCAP. Therefore, we will introduce a general stochastic model for the CCAP, and derive tight conditions to ensure asymptotic feasibility in a probabilistic sense. To solve the CCAP, we have proposed two different solution methods. Firstly, we have generalized the class of greedy heuristics proposed by Martello and Toth [88] for the GAP to the CCAP and we have proposed two local exchange procedures for improving a given (partial) solution for the CCAP. Secondly, we have generalized the Branch and Price scheme given by Savelsbergh [121] for the GAP to the CCAP and have studied the pricing problem for a particular subclass of CCAP's for which bounds can be efficiently found.

Throughout Parts II-IV our goal will be to analyze the behaviour of the solution methods proposed in Chapter 2. We will show asymptotic feasibility and optimality of some members of the class of greedy heuristics for the CCAP for the particular case of the GAP. Similar results will be found for many variants of the MPSSP. We will show that the Branch and Price scheme is another powerful tool when solving MPSSP's to optimality. Our main concern will be to identify subclasses of these optimization models for which the pricing problem can be solved efficiently.

The outline of the thesis is as follows. In Chapter 2 we introduce the class of CCAP's and present a general stochastic model for the CCAP and tight conditions to ensure asymptotic feasibility in a probabilistic sense. Moreover, we introduce a class of greedy heuristics and two local exchange procedures for improving a given (partial) solution for the CCAP, and a Branch and Price scheme together with the study of the pricing problem for a subclass of CCAP's. Part II is devoted to the study of the GAP which, as mentioned above, is a static model to evaluate two-level logistics distribution networks where production and storage take place at the same location. In Chapter 3 we show that the GAP belongs to the class of CCAP's, summarize the literature devoted to this problem, and we present some properties of the GAP which will be used in the rest of Part II. In Chapter 4 we analyze almost all the random generators proposed in the literature for the GAP and we numerically illustrate how the conclusions drawn about the performance of a greedy heuristic for the GAP differ depending on the random generator used. In Chapter 5 we show that, for large problem instances of the GAP generated by a general stochastic model, two greedy heuristics, defined by using some information of the LP-relaxation of the GAP, find a feasible and optimal solution with probability one. Part III is devoted to the study of a class of two-level MPSSP's where production and storage take place at the same location and only the production capacity is constrained. In Chapter 6 we introduce this class, we show that it is contained in the class of CCAP's, and we present some properties which will be used in the rest of this part. In Chapter 7, as for the GAP, we find explicit conditions to ensure asymptotic feasibility in the probabilistic sense for some variants of the MPSSP. In Chapter 8 we show that, for large problem instances of the MPSSP generated by a general stochastic model, a greedy heuristic, defined by using some information of the LP-relaxation of the MPSSP, finds a feasible and optimal solution with probability one for some variants of the MPSSP, as for the GAP. In Chapter 9 we analyze the pricing problem for the MPSSP and we propose a class of greedy heuristics to find feasible solutions for the pricing problem. In Part IV we extend the model proposed in Part III in two directions. In Chapter 10 we show that the MPSSP with three different types of additional constraints, namely throughput capacity, physical capacity and perishability constraints, can still be reformulated as a CCAP and we apply the results derived for the CCAP. In Chapter 11 we study three-level logistics distribution networks in which the plants and the warehouses have been decoupled, we show that these models can be *almost* reformulated as CCAP's, and that, except for the Branch and Price scheme, all the results derived for the CCAP still hold for these models. We end the thesis in Chapter 12 with a summary and some concluding remarks.

Chapter 2

A Class of Convex Capacitated Assignment Problems

2.1 Introduction

In a general assignment problem there are tasks which need to be processed and agents which can process these tasks. Each agent faces a set of capacity constraints and a cost when processing the tasks. Then the problem is how to assign each task to exactly one agent, so that the total cost of processing the tasks is minimized and each agent does not violate his capacity constraints. In the class of *Convex Capacitated Assignment Problems (CCAP)* each capacity constraint is linear with nonnegative coefficients and the costs are given by a convex function. The problem can be formulated as follows:

$$\text{minimize } \sum_{i=1}^m g_i(x_{i\cdot})$$

subject to

$$\begin{aligned} A^i x_{i\cdot} &\leq b_i & i = 1, \dots, m \\ \sum_{i=1}^m x_{ij} &= 1 & j = 1, \dots, n \\ x_{ij} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n \end{aligned}$$

where $g_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function, $A^i \in \mathcal{M}_{k_i \times n}$ is a nonnegative matrix, and $b_i \in \mathbb{R}^{k_i}$ is a nonnegative vector. Hereafter we will represent matrix A^i by its columns, i.e., $A^i = (A_1^i | \dots | A_n^i)$ where $A_j^i \in \mathbb{R}^{k_i}$ for each $j = 1, \dots, n$. The constraints associated with agent i , $A^i x_{i\cdot} \leq b_i$, define the feasible region of a multi-knapsack problem.

Other classes of assignment problems have been proposed in the literature. Ferland, Hertz and Lavoie [43] introduce a more general class of assignment problems, and show the applicability of object-oriented programming by developing software containing several heuristics. All set partitioning models discussed by Barnhart et al. [9] with convex and separable objective function in the index i are examples of convex capacitated assignment problems. They focus on branching rules and some computational issues relevant in the implementation of a Branch and Price scheme. In a similar context, Freling et al. [52] study the pricing problem of a class of convex assignment problems where the capacity constraints associated with each agent are defined by general convex sets.

The *Generalized Assignment Problem (GAP)* is one of the classical examples of a convex capacitated assignment problem (Ross and Soland [118]), where the cost function g_i associated with agent i is linear in x_i , and just one capacity constraint is faced by each agent, i.e., $k_i = 1$ for each $i = 1, \dots, m$. The GAP models the situation where a single resource available at the agents is consumed when processing tasks. Gavish and Pirkul [55] have studied a more general model, the Multi-Resource Generalized Assignment Problem (MRGAP) where several resources are available at the agents. This is still an example of a convex capacitated assignment problem where the objective function is linear as for the GAP, and each agent faces the same number of capacity constraints, i.e., $k_i = k$ for each $i = 1, \dots, m$. Ferland, Hertz and Lavoie [44] consider an exchange procedure for timetabling problems where the capacity constraints are not necessarily separable in the agents and the objective function is linear. Mazzola and Neebe [92] propose a Branch and Bound procedure for the particular situation where each agent must process exactly one task. This model can be seen as the classical Assignment Problem (see Nemhauser and Wolsey [100]) with side constraints.

Since the GAP is an \mathcal{NP} -Hard problem (see Martello and Toth [89]), so is the CCAP. Moreover, since the decision problem associated with the feasibility of the GAP is an \mathcal{NP} -Complete problem, so is the corresponding decision problem for the CCAP. Therefore, even to test whether a problem instance of the CCAP has at least one feasible solution is computationally hard. Hence, solving large problem instances to optimality may require a significant computational effort. The quality of the solution required and the technical limitations will determine the type of solution procedure to be used.

The outline of this chapter is as follows. In Section 2.2 we will introduce a general stochastic model for the class of convex capacitated assignment problems, and derive tight conditions to ensure asymptotic feasibility in a probabilistic sense when the number of tasks grows to infinity. In Section 2.3 we propose a class of greedy heuristics for the CCAP as well as two local exchange procedures for improving a given (partial) solution for the CCAP. In Section 2.4 each convex capacitated assignment problem is equivalently formulated as a set partitioning problem. In Section 2.5 we propose a Branch and Price procedure for solving this problem based on a column generation approach for the corresponding set partitioning formulation. This approach generalizes a similar Branch and Price procedure for the GAP (see Savelsbergh [121]). In Section 2.5.4 we study the pricing problem for a particular

subclass for which efficient bounds can be found. The chapter ends in Section 2.6 with a summary. Some of the results in this chapter can be found in Freling et al. [52].

2.2 Stochastic models and feasibility analysis

2.2.1 A general stochastic model

Solution procedures are meant to give answers to real-life problem instances. Generally, real data is only available for few scenarios. The limitation on the number of the data sets can bias the conclusions drawn about the behavior of the solution procedures. Thus, problem instances are generated to validate solution procedures. However, data generation schemes may also introduce biases into the computational results, as Hall and Posner [68] mention. They consider the feasibility of the problem instances an important characteristic of a data generation scheme. They propose two approaches to avoiding infeasible problem instances. The first one is to generate problem instances without regard for feasibility and discard the infeasible ones, and the second one is to enforce feasibility in the data generation process. Obviously, the first approach can be very time consuming when the generator of problem instances is not adequate.

Probabilistic analysis is a powerful tool when generating appropriate random data for a problem. Performing a feasibility analysis yields a suitable probabilistic model that can be used for randomly generating experimental data for the problem, with the property that the problem instances are asymptotically feasible with probability one. In this respect, Romeijn and Piersma [110] propose a stochastic model for the GAP where for each task the costs and requirements are i.i.d. random vectors, and the capacities depend linearly on the number of tasks. By means of empirical process theory, they derive a tight condition to ensure feasibility with probability one when the number of tasks goes to infinity. They also perform a value analysis of the GAP.

In the literature we mainly find probabilistic value analyses. They have been performed for a large variety of problems, starting with the pioneering paper by Beardwood, Halton and Hammersley [12] on a probabilistic analysis of Euclidean TSP's, spawning a vast number of papers on the probabilistic analysis of various variants of the TSP and VRP (see Bramel and Simchi-Levi [20] for an overview, and, for a more recent example, the probabilistic analysis of the inventory-routing problem by Chan, Federgruen and Simchi-Levi [28]).

Numerous other problems have also been analyzed probabilistically. Some well-known examples are a median location problem (Rhee and Talagrand [107]), the multi-knapsack problem (Van de Geer and Stougie [133]), a minimum flow time scheduling problem (Marchetti Spaccamela et al. [87]), the parallel machine scheduling problem (Piersma and Romeijn [104]), a generalized bin-packing problem (Federgruen and Van Ryzin [41]), the flow shop weighted completion time problem (Kaminisky and Simchi-Levi [73]), and the capacitated facility location problem (Piersma [103]). All these applications have in common that feasibility is not an issue since feasible problem instances can easily be characterized, so that a probabilistic analysis

of the optimal value of the optimization problem suffices.

As mentioned above, we will perform a feasibility analysis of the CCAP. Thus, when defining a stochastic model for the CCAP we can leave out parameters defining the objective function. Note that this does not preclude correlations between parameters. Consider the following probabilistic model for the parameters defining the feasible region of the CCAP. Let the random vectors $\mathbf{A}_j = ((\mathbf{A}_j^1)^\top, \dots, (\mathbf{A}_j^m)^\top)^\top$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $[\underline{A}, \overline{A}]^{k_1} \times \dots \times [\underline{A}, \overline{A}]^{k_m}$ where \underline{A} and $\overline{A} \in \mathbb{R}_+$. Furthermore, let b_i depend linearly on n , i.e., $b_i = \beta_i n$, for positive vectors $\beta_i \in \mathbb{R}^{k_i}$. Observe that m and k_i are fixed, thus the size of each problem instance only depends on the number of tasks n . Even though the requirements must be identically distributed, by considering the appropriate mixture of distribution functions, this stochastic model is suitable to model several types of tasks.

To analyze probabilistically the feasibility of the CCAP, we will use the same methodology as Romeijn and Piersma [110] for the GAP. They consider an auxiliary mixed integer linear problem to decide whether a given problem instance of the GAP has at least a feasible solution, or additional capacity is needed to ensure feasibility. A relationship between the optimal solution values of the auxiliary problem and its LP-relaxation is established. Through empirical process theory, the behaviour of the LP-relaxation is characterized. Altogether this yields a tight condition that ensures asymptotic feasibility with probability one when the number of tasks grows to infinity.

In the next section we will summarize the results used from empirical process theory.

2.2.2 Empirical processes

Let \mathcal{S} be a class of subsets of some space \mathbf{X} . For n distinct points x_1, \dots, x_n in \mathbf{X} , define

$$\Delta_{\mathcal{S}}(x_1, \dots, x_n) \equiv \text{card}(\mathcal{S} \cap \{x_1, \dots, x_n\} : S \in \mathcal{S}),$$

so, $\Delta_{\mathcal{S}}(x_1, \dots, x_n)$ counts the number of distinct subsets of $\{x_1, \dots, x_n\}$ that can be obtained when $\{x_1, \dots, x_n\}$ is intersected with sets in the class \mathcal{S} . Also define

$$m_{\mathcal{S}}(n) \equiv \sup\{\Delta_{\mathcal{S}}(x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbf{X}\}.$$

The class \mathcal{S} is called a *Vapnik-Chervonenkis class* (or *VC class*) if $m_{\mathcal{S}}(n) < 2^n$ for some $n \geq 1$.

For any subset \mathcal{Y} of \mathbb{R}^m , define the graph of a function $f : \mathcal{Y} \rightarrow \mathbb{R}$ by

$$\text{graph}(f) \equiv \{(s, t) \in \mathcal{Y} \times \mathbb{R} : 0 \leq t \leq f(s) \text{ or } f(s) \leq t \leq 0\}.$$

A class of real-valued functions is called a *Vapnik-Chervonenkis graph class* (or *VC graph class*) if the class of graphs of the functions is a VC class.

Theorem 2.2.1 (cf. Talagrand [128]) *Let X_1, X_2, \dots be a sequence of i.i.d. random variables taking values in a space (\mathbf{X}, \mathbb{A}) and let \mathcal{G} be a class of measurable real-valued functions on \mathbf{X} , such that*

- \mathcal{G} is a Vapnik-Chervonenkis graph class; and

- the functions in \mathcal{G} are uniformly bounded.

Then there exist ℓ and R such that, for all $n \geq 1$ and $\delta > 0$,

$$\Pr \left(\sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{j=1}^n g(x_j) - \mathcal{E}g(x_1) \right| > \delta \right) \leq \left(\frac{K\delta\sqrt{n}}{\ell R} \right)^\ell \cdot \exp \left(-\frac{2\delta^2 n}{R} \right)$$

where K is a universal constant not depending on (\mathbf{X}, \mathbb{A}) or \mathcal{G} .

A more extensive introduction to empirical processes can be found in Piersma [102] where probabilistic analyses are performed on several classical combinatorial problems.

2.2.3 Probabilistic feasibility analysis

The auxiliary problem (F) to characterize the feasibility of a convex capacitated assignment problem can be defined as

$$\begin{aligned} & \text{maximize } \xi \\ & \text{subject to} \end{aligned} \tag{F}$$

$$\begin{aligned} A^i x_i &\leq b_i - \xi e^{k_i} & i = 1, \dots, m \\ \sum_{i=1}^m x_{ij} &= 1 & j = 1, \dots, n \\ x_{ij} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n \\ \xi &\text{ free} \end{aligned}$$

where e^{k_i} denotes the vector in \mathbb{R}^{k_i} with all components equal to one. Note that this problem always has a feasible solution. Let v_n be the optimal value of (F), and v_n^{LP} be the optimal value of the LP-relaxation of (F). The convex capacitated assignment problem is feasible if and only if $v_n \geq 0$.

When empirical process theory is used to perform the probabilistic analysis, an essential task is to establish a relationship between the optimal solution values of the problem and its LP-relaxation. The following lemma shows that the values v_n and v_n^{LP} remain close, even if n grows large.

Lemma 2.2.2 *The difference between the optimal solution values of (F) and its LP-relaxation satisfies:*

$$v_n^{\text{LP}} - v_n \leq \left(\sum_{i=1}^m k_i \right) \cdot \bar{A}.$$

Proof: Rewrite the problem (F) with equality constraints and nonnegative variables only. We then obtain a problem with, in addition to the assignment constraints, $\sum_{i=1}^m k_i$ equality constraints. Now consider the optimal solution to the LP-relaxation

of (F). The number of variables having a nonzero value in this solution is no larger than the number of equality constraints in the reformulated problem. Since there is at least one nonzero assignment variable corresponding to each assignment constraint, and exactly one nonzero assignment variable corresponding to each assignment that is feasible with respect to the integrality constraints of (F), there can be no more than $\sum_{i=1}^m k_i$ assignments that are split. Converting the optimal LP-solution to a feasible solution to (F) by arbitrarily changing only those split assignments yields a solution to (F) that exceeds the LP-solution value by at most $(\sum_{i=1}^m k_i) \cdot \bar{A}$. Thus the desired inequality follows. \square

The following theorem uses empirical process theory to characterize the behaviour of the random variable \mathbf{v}_n^{LP} .

In the following, $\lambda = (\lambda_1^\top, \dots, \lambda_m^\top)^\top$ where $\lambda_i \in \mathbb{R}^{k_i}$.

Theorem 2.2.3 *There exist constants $\ell_{\mathcal{F}}$ and R_1 such that, for each $n \geq 1$ and $\delta > 0$,*

$$\Pr(|\tfrac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta| > \delta) \leq \left(\frac{K\delta\sqrt{n}}{\ell_{\mathcal{F}}R_1}\right)^{\ell_{\mathcal{F}}} \cdot \exp\left(-\frac{2\delta^2n}{R_1^2}\right)$$

where K is a universal constant,

$$\Delta = \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i^\top \beta_i - \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_1^i \right) \right)$$

and S is the unit simplex in $\mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_m}$.

Proof: Dualizing the capacity constraints in (F) with parameters $\lambda_i \in \mathbb{R}_+^{k_i}$, for each $i = 1, \dots, m$, yields the problem

$$\text{maximize } \xi + \sum_{i=1}^m \lambda_i^\top (b_i - \xi e^{k_i} - A^i x_i.)$$

subject to

$$\begin{aligned} \sum_{i=1}^m x_{ij} &= 1 & j = 1, \dots, n \\ x_{ij} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n \\ \xi &\text{ free.} \end{aligned}$$

Rearranging the terms in the objective function, we obtain

$$\begin{aligned} &\left(1 - \sum_{i=1}^m \lambda_i^\top e^{k_i}\right) \cdot \xi + \sum_{i=1}^m \lambda_i^\top b_i - \sum_{i=1}^m \lambda_i^\top A^i x_i. = \\ &= \left(1 - \sum_{i=1}^m \sum_{k=1}^{k_i} \lambda_{ik}\right) \cdot \xi + \sum_{i=1}^m \lambda_i^\top b_i - \sum_{i=1}^m \sum_{j=1}^n \lambda_i^\top A_j^i x_{ij}. \end{aligned}$$

Now let $v_n(\lambda)$ denote the optimal value of the relaxed problem. By strong duality, $v_n^{\text{LP}} = \min_{\lambda \geq 0} v_n(\lambda)$. First observe that if $\sum_{i=1}^m \sum_{k=1}^{k_i} \lambda_{ik} \neq 1$, then $v_n(\lambda) = \infty$. Thus, we can restrict the feasible region of the relaxed problem to the simplex S . But in this case the optimal solution of the relaxed problem is attained for

- $\xi = 0$; and
- $x_{ij} = 1$ if $i = \arg \min_{\nu=1, \dots, m} \lambda_\nu^\top A_j^\nu$ (where ties are broken arbitrarily), and $x_{ij} = 0$ otherwise.

Then we have that

$$\begin{aligned} v_n(\lambda) &= \sum_{i=1}^m \lambda_i^\top b_i - \sum_{j=1}^n \min_{i=1, \dots, m} \lambda_i^\top A_j^i \\ &= \sum_{j=1}^n \left(\sum_{i=1}^m \lambda_i^\top \beta_i - \min_{i=1, \dots, m} \lambda_i^\top A_j^i \right). \end{aligned}$$

Now define the function $f_\lambda : [\underline{A}, \overline{A}]^{k_1} \times \dots \times [\underline{A}, \overline{A}]^{k_m} \rightarrow \mathbb{R}$ by

$$f_\lambda(u) = \sum_{i=1}^m \lambda_i^\top \beta_i - \min_{i=1, \dots, m} \lambda_i^\top u_i$$

so that

$$v_n(\lambda) = \sum_{j=1}^n f_\lambda(A_j)$$

where A_j is a realization of the random vector \mathbf{A}_j . The function class $\mathcal{F} = \{f_\lambda : \lambda \in S\}$ is a VC graph class, since we can write

$$\begin{aligned} \text{graph}(f_\lambda) &= \\ &= \bigcup_{i=1}^m \left\{ (u, w) \in \mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_m} \times \mathbb{R} : 0 \leq w \leq \sum_{\nu=1}^m \lambda_\nu^\top \beta_\nu - \lambda_i^\top u_i \right\} \cup \\ &\quad \bigcap_{i=1}^m \left\{ (u, w) \in \mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_m} \times \mathbb{R} : 0 \geq w \geq \sum_{\nu=1}^m \lambda_\nu^\top \beta_\nu - \lambda_i^\top u_i \right\}. \end{aligned}$$

Moreover, this class is uniformly bounded, since

$$-\overline{A} \leq f_\lambda(u) \leq \sum_{i=1}^m \sum_{k=1}^{k_i} \beta_{ik}$$

for each $u \in [\underline{A}, \overline{A}]^{k_1} \times \dots \times [\underline{A}, \overline{A}]^{k_m}$. Noting that

$$\begin{aligned} \left| \frac{1}{n} v_n^{\text{LP}} - \Delta \right| &= \left| \frac{1}{n} \min_{\lambda \in S} \sum_{j=1}^n f_\lambda(\mathbf{A}_j) - \min_{\lambda \in S} \mathcal{E} f_\lambda(\mathbf{A}_1) \right| \\ &\leq \sup_{\lambda \in S} \left| \frac{1}{n} \sum_{j=1}^n f_\lambda(\mathbf{A}_j) - \mathcal{E} f_\lambda(\mathbf{A}_1) \right|, \end{aligned}$$

the result now follows directly from Theorem 2.2.1. \square

The bound on the tail probability given in the last theorem will be used to show a tight condition to ensure feasibility with probability one when n grows to infinity. This is a generalization of Theorem 3.2 from Romeijn and Piersma [110].

Theorem 2.2.4 *As $n \rightarrow \infty$, the CCAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$.*

Proof: Recall that the CCAP is feasible if and only if \mathbf{v}_n is nonnegative. Therefore,

$$\begin{aligned} \Pr(\text{the CCAP is feasible}) &= \Pr(\mathbf{v}_n \geq 0) \\ &\leq \Pr(\mathbf{v}_n^{\text{LP}} \geq 0) \\ &= \Pr\left(\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta \geq -\Delta\right) \\ &\leq \Pr\left(\left|\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta\right| \geq -\Delta\right). \end{aligned}$$

It follows that the CCAP is feasible with probability zero (and thus infeasible with probability one) if $\Delta < 0$ since then, for $0 < \varepsilon < -\Delta$,

$$\begin{aligned} \Pr\left(\left|\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta\right| \geq -\Delta\right) &\leq \Pr\left(\left|\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta\right| > \varepsilon\right) \\ &\leq \left(\frac{K\varepsilon\sqrt{n}}{\ell_{\mathcal{F}}R_1}\right)^{\ell_{\mathcal{F}}} \cdot \exp\left(-\frac{2\varepsilon^2n}{R_1^2}\right) \end{aligned}$$

by Theorem 2.2.4, and

$$\sum_{n=1}^{\infty} \left(\frac{K\varepsilon\sqrt{n}}{\ell_{\mathcal{F}}R_1}\right)^{\ell_{\mathcal{F}}} \cdot \exp\left(-\frac{2\varepsilon^2n}{R_1^2}\right) < \infty.$$

Similarly, by using Lemma 2.2.2

$$\begin{aligned} \Pr(\text{the CCAP is infeasible}) &= \Pr(\mathbf{v}_n < 0) \\ &\leq \Pr\left(\mathbf{v}_n^{\text{LP}} - \left(\sum_{i=1}^m k_i\right) \cdot \bar{A} < 0\right) \\ &= \Pr\left(\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta < \left(\sum_{i=1}^m k_i\right) \cdot \bar{A}/n - \Delta\right) \\ &\leq \Pr\left(\left|\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta\right| > \Delta - \left(\sum_{i=1}^m k_i\right) \cdot \bar{A}/n\right). \end{aligned}$$

It follows that the CCAP is infeasible with probability zero (and thus feasible with probability one) if $\Delta > 0$ since then, for $n > (\sum_{i=1}^m k_i) \cdot \bar{A}/\Delta$,

$$\begin{aligned} \Pr\left(\left|\frac{1}{n}\mathbf{v}_n^{\text{LP}} - \Delta\right| > \Delta - \left(\sum_{i=1}^m k_i\right) \cdot \bar{A}/n\right) &\leq \\ &\leq \left(\frac{K(\Delta n - (\sum_{i=1}^m k_i) \cdot \bar{A})}{\ell_{\mathcal{F}}R_1\sqrt{n}}\right)^{\ell_{\mathcal{F}}} \cdot \exp\left(-\frac{2(\Delta n - (\sum_{i=1}^m k_i) \cdot \bar{A})^2}{nR_1^2}\right) \end{aligned}$$

by Theorem 2.2.4, and

$$\sum_{n > (\sum_{i=1}^m k_i) \cdot \bar{A} / \Delta} \left(\frac{K(\Delta n - (\sum_{i=1}^m k_i) \cdot \bar{A})}{\ell_{\mathcal{F}} R_1 \sqrt{n}} \right)^{\ell_{\mathcal{F}}} \cdot \exp \left(- \frac{2(\Delta n - (\sum_{i=1}^m k_i) \cdot \bar{A})^2}{n R_1^2} \right) < \infty.$$

□

For the case of the GAP, this result reduces to the one derived by Romeijn and Piersma [110]. This is an implicit condition which, most of the times, is difficult to check. They were only able to find explicit conditions when the requirements were agent-independent, i.e. $A_j^i = D_j$ for each $i = 1, \dots, m$. In this case, the problem defining the value of Δ turns into a linear one. (Recall that for the GAP A_j^i are scalars.) So, the minimization can be reduced to inspecting the extreme points of the unit simplex in \mathbb{R}^m . For a general convex capacitated assignment problem, the presence of the multi-knapsack constraints for each agent make it impossible to use the same reasoning.

In Chapter 7 we will encounter special cases of the CCAP for which the feasibility condition that $\Delta > 0$ can also be made explicit. However, for some of the variants of these CCAP's, this stochastic model is not suitable since it imposes independence between the requirement vectors. In the following section, we will extend the stochastic model for the CCAP given in Section 2.2.1 to be able to deal also with these cases.

2.2.4 Extension of the stochastic model

The stochastic model for the CCAP given in Section 2.2.1 assumes that the vector of requirements of the tasks \mathbf{A}_j ($j = 1, \dots, n$) are independently distributed. In Chapter 7 we will see that this condition is not fulfilled in some examples of CCAP's. In this section, we relax the independence assumption in the following way. We will generate the requirement parameters for subsets of tasks, so that the vectors of requirement parameters corresponding to different subsets are independently and identically distributed. Note that we do not impose any condition on the requirements of the tasks within the same subset. More precisely, let J_j be the j -th subset of tasks and, for now, assume that $|J_j| = J$ for each $j = 1, \dots, n$. Observe that the number of tasks in the CCAP formulation is now equal to nJ . For each $j = 1, \dots, n$ and $\ell = 1, \dots, J$, let $\mathbf{A}_{J_j \ell}$ denote the vector of requirements of the ℓ -th task in the subset J_j . Let the random vectors $\mathbf{A}_{J_j} = (\mathbf{A}_{J_j \ell}^\top)_{\ell \in J_j}^\top$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $\prod_{\ell=1}^J ([\underline{A}, \bar{A}]^{k_1} \times \dots \times [\underline{A}, \bar{A}]^{k_m})$ where \underline{A} and $\bar{A} \in \mathbb{R}_+$. As before, let b_i depend linearly on n , i.e., $b_i = \beta_i n$, for positive vectors $\beta_i \in \mathbb{R}^{k_i}$.

In the following theorem we show a similar condition as in Theorem 2.2.4 to ensure feasibility with probability one when n grows to infinity. In this case the

excess capacity reads as follows:

$$\Delta = \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i^\top \beta_i - \sum_{\ell=1}^J \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_{J_1 \ell}^i \right) \right)$$

where S is the unit simplex in $\mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_m}$.

Theorem 2.2.5 *As $n \rightarrow \infty$, the CCAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$.*

Proof: From the proof of Theorem 2.2.4, it is enough to show that there exist constants ℓ and R such that, for each $n \geq 1$ and $\delta > 0$, we have that

$$\Pr \left(\left| \frac{1}{n} \mathbf{v}_n^{\text{LP}} - \Delta \right| > \delta \right) \leq \left(\frac{K\delta\sqrt{n}}{\ell R} \right)^\ell \cdot \exp \left(-\frac{2\delta^2 n}{R^2} \right).$$

Following similar steps as in the proof of Theorem 2.2.3, we have that $\mathbf{v}_n^{\text{LP}} = \min_{\lambda \geq 0} v_n(\lambda)$ where

$$v_n(\lambda) = \sum_{j=1}^n \left(\sum_{i=1}^m \lambda_i^\top \beta_i - \sum_{\ell=1}^J \min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_{J_j \ell}^i \right).$$

Now define the function $f_\lambda : \prod_{\ell=1}^J ([\underline{A}, \overline{A}]^{k_1} \times \dots \times [\underline{A}, \overline{A}]^{k_m}) \rightarrow \mathbb{R}$ by

$$f_\lambda(u) = \sum_{i=1}^m \lambda_i^\top \beta_i - \sum_{\ell=1}^J \min_{i=1, \dots, m} \lambda_i^\top u_{i\ell}$$

so that

$$v_n(\lambda) = \sum_{j=1}^n f_\lambda(\mathbf{A}_{J_j})$$

where \mathbf{A}_{J_j} is a realization of the random vector \mathbf{A}_{J_j} . With the same arguments as in Theorem 2.2.3, we can show that the function class $\mathcal{F} = \{f_\lambda : \lambda \in S\}$ is a VC graph class and uniformly bounded, and the result follows similarly to Theorem 2.2.3. \square

A similar result can be obtained when the size of the subsets generated is not unique but it can take κ values, say J^s , for each $s = 1, \dots, \kappa$. We will assume that the size of the subsets of tasks is multinomial-distributed with parameters $(\pi_1, \dots, \pi_\kappa)$, with $\pi_s \geq 0$ and $\sum_{s=1}^\kappa \pi_s = 1$, where π_s is the probability of generating a subset of J^s tasks. Observe that the number of tasks in the CCAP formulation is random and equal to $\sum_{j=1}^n \mathbf{J}_j$. For each $j = 1, \dots, n$, $s = 1, \dots, \kappa$, and $\ell = 1, \dots, J$, let $\mathbf{A}_{J_j s \ell}$ denote the vector of requirements of the ℓ -th task in the subset J_j when $|J_j| = J^s$. Let the random vectors $\mathbf{A}_{J_j} = ((\mathbf{A}_{J_j 1 \ell}^\top)_{\ell \in J_j}^\top, \dots, (\mathbf{A}_{J_j \kappa \ell}^\top)_{\ell \in J_j}^\top)^\top$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $\prod_{s=1}^\kappa \prod_{\ell=1}^{J^s} ([\underline{A}, \overline{A}]^{k_1} \times \dots \times [\underline{A}, \overline{A}]^{k_m})$ where \underline{A} and $\overline{A} \in \mathbb{R}_+$. Note that only one of the vector of requirements, say $(\mathbf{A}_{J_j s \ell}^\top)_{\ell \in J_j}^\top$, will be in effect for

subset J_j , depending on the realized number of tasks in that subset. As before, let b_i depend linearly on n , i.e., $b_i = \beta_i n$, for positive vectors $\beta_i \in \mathbb{R}^{k_i}$.

In this case, the excess capacity reads

$$\Delta = \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i^\top \beta_i - \sum_{s=1}^{\kappa} \pi_s \sum_{\ell=1}^{J_s} \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_{J_1 s \ell}^i \right) \right)$$

where S is the unit simplex in $\mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_m}$. Then, the tight condition for feasibility reads as follows.

Theorem 2.2.6 *As $n \rightarrow \infty$, the CCAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$.*

Proof: Similar to the proof of Theorem 2.2.5. □

2.2.5 Asymptotically equivalent stochastic models

The stochastic model for the CCAP introduced above considers deterministic right-hand sides. This excludes, for example, cases where right-hand sides depend on the requirements of the tasks which is widely used in random generators for the GAP, as will be shown in Chapter 4. Stochastic models with random right-hand sides where the relative capacities in the infinity are constant can be still analyzed by means of the concept of asymptotically equivalent stochastic models. It can be shown that feasibility conditions are the same for asymptotically equivalent stochastic models. Therefore, we can obtain a similar result to Theorem 2.2.4. Hereafter (\mathbf{A}, \mathbf{b}) will denote a stochastic model for the parameters of the feasible region of the CCAP where $\mathbf{A} = (\mathbf{A}_j)$, \mathbf{A}_j represents the vector of requirements for task j , $\mathbf{b} = (\mathbf{b}_i)$ and \mathbf{b}_i is the right-hand side vector associated with agent i .

Definition 2.2.7 *Let (\mathbf{A}, \mathbf{b}) and $(\mathbf{A}', \mathbf{b}')$ be two stochastic models for the feasible region of the CCAP. We will say that (\mathbf{A}, \mathbf{b}) and $(\mathbf{A}', \mathbf{b}')$ are asymptotically equivalent if the following statements hold:*

1. *Requirements are equally distributed in both of the models.*
2. *For each $i = 1, \dots, m$ and $k = 1, \dots, k_i$, it holds*

$$\frac{\mathbf{b}_{ik} - \mathbf{b}'_{ik}}{n} \rightarrow 0$$

with probability one when n goes to infinity.

In the next result we show that feasibility conditions are the same for asymptotically equivalent stochastic models.

Proposition 2.2.8 *Let (\mathbf{A}, \mathbf{b}) and $(\mathbf{A}', \mathbf{b}')$ be two asymptotically equivalent stochastic models for the parameters defining the feasible region of the CCAP. Then, (\mathbf{A}, \mathbf{b}) is feasible with probability one when n goes to infinity if and only if $(\mathbf{A}', \mathbf{b}')$ is feasible with probability one when n goes to infinity.*

Proof: Let \mathbf{v}_n^{LP} be the random variable defined as the optimal value of the LP-relaxation of (F) for the stochastic model (\mathbf{A}, \mathbf{b}) and $\mathbf{v}'_n^{\text{LP}}$ for $(\mathbf{A}', \mathbf{b}')$. It is enough to prove that

$$\frac{1}{n} |\mathbf{v}_n^{\text{LP}} - \mathbf{v}'_n^{\text{LP}}| \rightarrow 0$$

with probability one when n goes to infinity. From the proof of Theorem 2.2.3, we know that

$$\mathbf{v}_n^{\text{LP}} = \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i^\top \mathbf{b}_i - \sum_{j=1}^n \min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_j^i \right),$$

and similarly for $\mathbf{v}'_n^{\text{LP}}$. So,

$$\begin{aligned} \frac{1}{n} |\mathbf{v}_n^{\text{LP}} - \mathbf{v}'_n^{\text{LP}}| &= \\ &= \frac{1}{n} \left| \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i^\top \mathbf{b}_i - \sum_{j=1}^n \min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_j^i \right) \right. \\ &\quad \left. - \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i^\top \mathbf{b}'_i - \sum_{j=1}^n \min_{i=1, \dots, m} \lambda_i^\top (\mathbf{A}')_j^i \right) \right| \\ &\leq \frac{1}{n} \sup_{\lambda \in S} \left| \left(\sum_{i=1}^m \lambda_i^\top \mathbf{b}_i - \sum_{j=1}^n \min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_j^i \right) \right. \\ &\quad \left. - \left(\sum_{i=1}^m \lambda_i^\top \mathbf{b}'_i - \sum_{j=1}^n \min_{i=1, \dots, m} \lambda_i^\top (\mathbf{A}')_j^i \right) \right| \\ &\leq \frac{1}{n} \sup_{\lambda \in S} \left| \sum_{i=1}^m \lambda_i^\top (\mathbf{b}_i - \mathbf{b}'_i) \right| \\ &\quad + \frac{1}{n} \sup_{\lambda \in S} \sum_{j=1}^n \left| \min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_j^i - \min_{i=1, \dots, m} \lambda_i^\top (\mathbf{A}')_j^i \right| \\ &\leq \frac{1}{n} \sup_{\lambda \in S} \sum_{i=1}^m |\lambda_i^\top (\mathbf{b}_i - \mathbf{b}'_i)| + \frac{1}{n} \sup_{\lambda \in S} \sum_{j=1}^n \max_{i=1, \dots, m} |\lambda_i^\top (\mathbf{A}_j^i - (\mathbf{A}')_j^i)| \\ &\leq \frac{1}{n} \sup_{\lambda \in S} \sum_{i=1}^m \sum_{k=1}^{k_i} \lambda_{ik} |\mathbf{b}_{ik} - \mathbf{b}'_{ik}| + \frac{1}{n} \sup_{\lambda \in S} \sum_{j=1}^n \max_{i=1, \dots, m} \sum_{k=1}^{k_i} \lambda_{ik} |\mathbf{A}_{jk}^i - (\mathbf{A}')_{jk}^i| \\ &\leq \frac{1}{n} \sum_{i=1}^m \sum_{k=1}^{k_i} |\mathbf{b}_{ik} - \mathbf{b}'_{ik}| + \frac{1}{n} \sum_{j=1}^n \max_{i=1, \dots, m} \sum_{k=1}^{k_i} |\mathbf{A}_{jk}^i - (\mathbf{A}')_{jk}^i|, \end{aligned}$$

where the last inequality follows since $\lambda_{ik} \leq 1$ for each $i = 1, \dots, m$ and $k = 1, \dots, k_i$. When n goes to infinity, the first term goes to zero with probability one by applying Claim 2 of the definition of asymptotically equivalent stochastic models. Similarly,

by using the Law of the Large Numbers and Claim 1 from the same definition we can show that the second term goes to zero when n goes to infinity with probability one. Hence, the desired result follows. \square

2.3 Heuristic solution approaches

2.3.1 A class of greedy heuristics for the CCAP

In Section 2.1 was shown that the CCAP is an \mathcal{NP} -Hard problem. Therefore, solving large instances of the problem to optimality can require a substantial computational effort. This calls for heuristic approaches where good solutions can be found with a reasonable computational effort. There clearly are situations where a good solution is sufficient in its own right. But in addition, a good suboptimal solution can accelerate an exact procedure.

Martello and Toth [88] propose a class of greedy heuristics widely used for the GAP. In Chapter 5, we will show asymptotic feasibility and optimality for two elements of this class. Expecting a similar success, we have generalized this class of greedy heuristics for the class of convex capacitated assignment problems. In Chapter 8, we will show asymptotic feasibility and optimality of one of those greedy heuristics for a member of the class of CCAP's.

The basic idea of this greedy heuristic is that each possible assignment of a task to an agent is evaluated by a pseudo-cost function $f(i, j)$. The *desirability* of assigning a task is measured by the difference between the second smallest and the smallest values of $f(i, j)$. Assignments of tasks to their best agents are made in decreasing order of this difference. Along the way, some agents will not be able to handle some of the tasks due to the constraints faced by each agent, and consequently the values of the desirabilities will be updated taking into account that the two most desirable agents for each task should be feasible.

We will denote a partial solution for the CCAP by x^G . Let \hat{j} be a task which has not been assigned yet and $x^G \cup \{(i, \hat{j})\}$ the partial solution for the CCAP where the assignment of task \hat{j} to agent \hat{i} is added to x^G . More formally,

$$(x^G \cup \{(i, \hat{j})\})_{ij} = \begin{cases} x_{ij}^G & \text{if } j \neq \hat{j}; i = 1, \dots, m \\ 1 & \text{if } (i, j) = (\hat{i}, \hat{j}) \\ 0 & \text{otherwise.} \end{cases}$$

This greedy heuristic can formally be written as follows:

Greedy heuristic for the CCAP

Step 0. Set $\mathcal{L} = \{1, \dots, n\}$, $\mathcal{NA} = \emptyset$, and $x_{ij}^G = 0$ for each $i = 1, \dots, m$ and $j = 1, \dots, n$.

Step 1. Let

$$\mathcal{F}_j = \{i = 1, \dots, m : A^i(x^G \cup \{(i, j)\})_i \leq b_i\} \quad \text{for } j \in \mathcal{L}.$$

If $\mathcal{F}_j = \emptyset$ for some $j \in \mathcal{L}$, the algorithm cannot assign task j ; then set $\mathcal{L} = \mathcal{L} \setminus \{j\}$, $\mathcal{NA} = \mathcal{NA} \cup \{j\}$, and repeat Step 1. Otherwise, let

$$\begin{aligned} i_j &\in \arg \min_{i \in \mathcal{F}_j} f(i, j) && \text{for } j \in \mathcal{L} \\ \rho_j &= \min_{\substack{s \in \mathcal{F}_j \\ s \neq i_j}} f(s, j) - f(i_j, j) && \text{for } j \in \mathcal{L}. \end{aligned}$$

Step 2. Let $\hat{j} \in \arg \max_{j \in \mathcal{L}} \rho_j$. Set

$$\begin{aligned} x_{i_{\hat{j}}\hat{j}}^G &= 1 \\ \mathcal{L} &= \mathcal{L} \setminus \{\hat{j}\}. \end{aligned}$$

Step 3. If $\mathcal{L} = \emptyset$: STOP. If $\mathcal{NA} = \emptyset$, x^G is a feasible solution for the CCAP, otherwise x^G is a partial feasible solution for the CCAP. Otherwise, go to Step 1.

The output of this heuristic is a vector of feasible assignments x^G , which is (at least) a partial solution to the CCAP. The challenge is to specify a pseudo-cost function that will yield a good (or at least a feasible) solution to the CCAP. Martello and Toth [88] have suggested four pseudo-cost functions for the GAP. In the following chapters we will investigate choices which asymptotically yield a feasible and optimal solution with probability one.

Note that, by not assigning the task with the largest difference between the two smallest values for the corresponding pseudo-cost function in a greedy fashion, but rather choosing the task to be assigned randomly among a list of candidates having the largest differences, a so-called GRASP algorithm can easily be constructed (see e.g. Feo and Resende [42] for an overview of GRASP algorithms).

This greedy heuristic does not guarantee that a feasible solution will always be found. In the worst case, the heuristic provides a partial solution for the CCAP which means that capacity constraints are not violated, but there may exist tasks which are not assigned to any agent. In the following section, we will describe two local exchange procedures to improve the current solution for the CCAP. The first one tries to assign the tasks where the heuristic failed, i.e., those ones in the set \mathcal{NA} . The second local exchange procedure tries to improve the objective value of the current solution for the CCAP.

Both of the procedures are based on 2-exchanges of assigned tasks. Recall that x^G is the current partial solution of the CCAP, \mathcal{NA} the set of non-assigned tasks in x^G , and for each $\ell \notin \mathcal{NA}$, i_ℓ the agent to which task ℓ is assigned in x^G , i.e., $x_{i_\ell \ell}^G = 1$. Let ℓ and p two assigned tasks, i.e., ℓ and $p \notin \mathcal{NA}$ so that $i_\ell \neq i_p$, and $x^G \otimes \{(\ell, p)\}$ denote the partial solution for the CCAP where the assignments of tasks ℓ and p in x^G are exchanged. More formally,

$$(x^G \otimes \{(\ell, p)\})_{ij} = \begin{cases} x_{ij}^G & \text{if } j \neq \ell, p; i = 1, \dots, m \\ x_{ip}^G & \text{if } j = \ell; i = 1, \dots, m \\ x_{i\ell}^G & \text{if } j = p; i = 1, \dots, m. \end{cases}$$

2.3.2 Improving the current solution

Improving feasibility

Different primal heuristics have been developed to construct feasible solutions for the GAP from a given solution. Mainly, we can mention those ones suggested by Jörnsten and Näsberg [71], Guignard and Rosenwein [66], and Lorena and Narciso [85]. These procedures are based on local exchanges. They start with a solution that violates capacity constraints or that does not assign all the tasks properly, i.e., there are non-assigned or multiply-assigned tasks. Observe that multiply-assigned tasks violate integrality constraints.

We propose a procedure for the CCAP based also on local exchanges. As mentioned above, the heuristic provides a partial solution for the CCAP satisfying the capacity constraints and with few non-assigned tasks, and we try to assign them by creating available capacity with the exchange of two assigned tasks. Given a non-assigned task j , its possible assignment to agent i is measured by $r(i, j)$. The *best* agent, say i_j , is defined as the one minimizing $r(i, j)$. The heuristic will look for a task ℓ assigned to agent i_j and a task p assigned to an agent different from i_j , such that by exchanging ℓ and p we still have a partial solution for the CCAP. Moreover, this exchange should yield additional available capacity at agent i_j to assign task j to agent i_j .

Given a non-assigned task $j \in \mathcal{NA}$ and an agent i_j , we will say that (ℓ, p) is a feasible exchange for assignment (i_j, j) if task ℓ is assigned to agent i_j , $i_\ell = i_j$, and capacity constraints are not violated when we assign task j to agent i_j and we exchange the assignments of tasks ℓ and p with $i_j \neq i_p$, in other words, when

$$A^i((x^G \cup \{(i_j, j)\}) \otimes \{(\ell, p)\})_i \leq b_i$$

for each $i = 1, \dots, m$.

Our local exchange procedure to improve feasibility proceeds as follows. The elements of \mathcal{NA} are assigned to their most desirable agent in decreasing order of $r(i_j, j)$, either directly when agent i_j has sufficient capacity available, or through a feasible exchange, if one can be found. If none of those attempts succeeds, we redefine the most desirable agent of j by deleting agent i_j from consideration for that task. We repeat this procedure until task j is assigned or there is no agent to assign to it. The local exchange procedure that we implemented was the following.

Improving feasibility

Step 0. Set $\mathcal{L} = \mathcal{NA}$ and $\mathcal{F}_j = \{1, \dots, m\}$ for each $j \in \mathcal{L}$.

Step 1. If $\mathcal{F}_j = \emptyset$ for some $j \in \mathcal{L}$, the algorithm cannot assign task j , then $\mathcal{L} = \mathcal{L} \setminus \{j\}$ and repeat Step 1. Otherwise, let

$$i_j = \arg \min_{i \in \mathcal{F}_j} r(i, j) \quad \text{for } j \in \mathcal{L}$$

$$o_j = r(i, j) \quad \text{for } j \in \mathcal{L}.$$

Step 2. Let $\hat{j} = \arg \max_{j \in \mathcal{L}} \varrho_j$. If

$$A^{i_{\hat{j}}}(x^G \cup \{(i_{\hat{j}}, \hat{j})\})_{i_{\hat{j}}} \leq b_{i_{\hat{j}}},$$

assign task \hat{j} to agent $i_{\hat{j}}$, i.e.,

$$\begin{aligned} x_{i_{\hat{j}}\hat{j}}^G &= 1 \\ \mathcal{L} &= \mathcal{L} \setminus \{\hat{j}\} \\ \mathcal{NA} &= \mathcal{NA} \setminus \{\hat{j}\}, \end{aligned}$$

and go to Step 1.

Step 3. Look for tasks ℓ and $p \notin \mathcal{NA}$ such that (ℓ, p) is a feasible exchange for assignment $(i_{\hat{j}}, \hat{j})$. If there is no feasible exchange, make agent $i_{\hat{j}}$ infeasible for task \hat{j} , i.e., $\mathcal{F}_{\hat{j}} = \mathcal{F}_{\hat{j}} \setminus \{i_{\hat{j}}\}$ and go to Step 1. Otherwise, exchange the assignments of tasks ℓ and p , and assign task \hat{j} to agent $i_{\hat{j}}$, i.e.,

$$\begin{aligned} x_{i_{\hat{j}}p}^G &= 1 \\ x_{i_{\hat{j}}\ell}^G &= 0 \\ x_{i_p\ell}^G &= 1 \\ x_{i_pp}^G &= 0 \\ x_{i_{\hat{j}}\hat{j}}^G &= 1 \\ \mathcal{L} &= \mathcal{L} \setminus \{\hat{j}\} \\ \mathcal{NA} &= \mathcal{NA} \setminus \{\hat{j}\}. \end{aligned}$$

Step 4. If $\mathcal{L} \neq \emptyset$, go to Step 1. Otherwise, STOP. If $\mathcal{NA} = \emptyset$, x^G is a feasible solution for the CCAP. Otherwise, x^G is a partial solution for the CCAP.

Improving the solution value

Finally, we attempt to improve the objective value of the current solution for the CCAP. Again, some heuristics can be found in the literature to improve the objective value for the GAP. Trick [131] observes that fixing the assignment of a subset of tasks still results in a GAP. He proposes to randomly fix a part of the current solution and to solve the rest using his LR-heuristic (see Section 3.3). However, the most widely used heuristics to improve the objective function are based on local exchanges. Martello and Toth [88] propose a 1-exchange where a task is reassigned when an improvement on the objective function can be achieved feasibly. Jörnsten and Näsberg [71] propose a 1-exchange and a 2-exchange. More detailed explanation about the 2-exchange procedure for the GAP is given by Wilson [136].

We perform a sequence of improving exchanges between pairs of assignments to improve the objective value of the current solution for the CCAP. We say that (ℓ, p) is an improving exchange if $i_{\ell} \neq i_p$ and

$$A^i(x^G \otimes \{(\ell, p)\})_i \leq b_i \quad i = 1, \dots, m$$

$$\sum_{i=1}^m g_i((x^G \otimes \{(\ell, p)\})_{i.}) < \sum_{i=1}^m g_i(x_{i.}^G),$$

which means that after exchanging the assignment of tasks ℓ and p we still have a feasible solution for the CCAP, but with lower objective value. The order in which these exchanges are considered is either the natural, lexicographic ordering, or an ordering based on the pseudo-cost function used in the greedy heuristic (see Section 2.3.1). In particular, pairs of tasks (ℓ, p) are considered for exchange in decreasing order of

$$(f(i_\ell, \ell) + f(i_p, p)) - (f(i_\ell, p) + f(i_p, \ell))$$

where i_ℓ and i_p are the agents to which tasks ℓ and p are assigned in the current solution.

Improving objective value

Step 0. Let $\mathcal{I} = \{(\ell, p) : \ell, p = 1 \dots, m; \ell < p\}$ and we assume that the elements of \mathcal{I} are ordered. Set $\kappa = 0$.

Step 1. Let $(\hat{\ell}, \hat{p})$ be the first element of \mathcal{I} and $\mathcal{I} = \mathcal{I} \setminus \{(\hat{\ell}, \hat{p})\}$. If

$$\begin{aligned} A^i(x^G \otimes \{(\hat{\ell}, \hat{p})\})_{i.} &\leq b_i & i = 1, \dots, m \\ \sum_{i=1}^m g_i((x^G \otimes \{(\hat{\ell}, \hat{p})\})_{i.}) &< \sum_{i=1}^m g_i(x_{i.}^G), \end{aligned}$$

then exchange the assignments of tasks $\hat{\ell}$ and \hat{p} , i.e.,

$$\begin{aligned} x_{i_{\hat{\ell}}p}^G &= 1 \\ x_{i_{\hat{\ell}}\ell}^G &= 0 \\ x_{i_{\hat{p}}\ell}^G &= 1 \\ x_{i_{\hat{p}}p}^G &= 0 \\ \kappa &= \kappa + 1. \end{aligned}$$

Step 2. If $\mathcal{I} \neq \emptyset$: go to Step 1. Otherwise, if $\kappa > 0$ go to Step 0. Otherwise, STOP.

We may observe that this local exchange procedure stops when no more improving exchanges can be found.

2.4 A set partitioning formulation

The convex capacitated assignment problem can be formulated as a set partitioning problem, in a similar way as done for the GAP by Cattrysse, Salomon and Van

Wassenhove [25]; and Savelsbergh [121]. In particular, a feasible solution for the CCAP can be seen as a partition of the set of tasks $\{1, \dots, n\}$ into m subsets. Each element of the partition is associated with one of the m agents.

Let L_i be the number of subsets of tasks that can feasibly be assigned to agent i ($i = 1, \dots, m$). Let α_i^ℓ denote the ℓ -th subset (for fixed i), i.e., $\alpha_{ij}^\ell = 1$ if task j is an element of subset ℓ for agent i , and $\alpha_{ij}^\ell = 0$ otherwise. We will call α_i^ℓ the ℓ -th *column* for agent i . The vector α_i^ℓ is a zero-one feasible solution of the multi-knapsack constraints associated with agent i . Then, the set partitioning problem can be formulated as follows:

$$\text{minimize } \sum_{i=1}^m \sum_{\ell=1}^{L_i} g_i(\alpha_i^\ell) y_i^\ell$$

subject to (MP)

$$\sum_{i=1}^m \sum_{\ell=1}^{L_i} \alpha_{ij}^\ell y_i^\ell = 1 \quad j = 1, \dots, n \quad (2.1)$$

$$\begin{aligned} \sum_{\ell=1}^{L_i} y_i^\ell &= 1 & i = 1, \dots, m \\ y_i^\ell &\in \{0, 1\} & \ell = 1, \dots, L_i; i = 1, \dots, m \end{aligned} \quad (2.2)$$

where y_i^ℓ is equal to 1 if column ℓ is chosen for agent i , and 0 otherwise. As mentioned by Barnhart et al. [9], the convexity constraint (2.2) for agent i ($i = 1, \dots, m$) can be written as

$$\sum_{\ell=1}^{L_i} y_i^\ell \leq 1$$

if $\alpha_{ij} = 0$ for each $j = 1, \dots, n$ is a feasible column for agent i with associated costs $g_i(\alpha_i) = 0$. One of the advantages of (MP) is that its objective function is linear while the one of the CCAP is, in general, a convex function. Furthermore each feasible solution for its linear relaxation LP(MP) is a convex combination of zero-one solutions of the multi-knapsack constraints associated with agent i . Therefore, LP(MP) gives a bound on the optimal solution value of (MP) that is at least as tight (and usually tighter) as the one obtained by relaxing the integrality constraints in the CCAP, R(CCAP). Hence, if we let $v(\text{R(CCAP)})$ and $v(\text{LP(MP)})$ denote the optimal objective values of R(CCAP) and LP(MP), respectively, then the following holds.

Proposition 2.4.1 *The following inequality holds:*

$$v(\text{R(CCAP)}) \leq v(\text{LP(MP)}).$$

Proof: First of all, note that if LP(MP) is infeasible, the inequality follows directly since in that case $v(\text{LP(MP)}) = \infty$. In the more interesting case that LP(MP) is feasible, the desired inequality follows from convexity arguments. We may observe

that both relaxations can be obtained by relaxing the integrality constraints to non-negativity constraints. Each feasible solution to LP(MP) can be transformed to a feasible solution to R(CCAP) as follows:

$$x_{ij} = \sum_{\ell=1}^{L_i} \alpha_{ij}^{\ell} y_i^{\ell} \quad i = 1, \dots, m; j = 1, \dots, n.$$

For each $i = 1, \dots, m$, vector x_i is a convex combination of vectors α_i^{ℓ} for $\ell = 1, \dots, L_i$. Since all constraints in R(CCAP) are linear so convex, x is a feasible solution for R(CCAP). Moreover, by convexity of the functions g_i we have that

$$\sum_{i=1}^m g_i(x_i) = \sum_{i=1}^m g_i \left(\sum_{\ell=1}^{L_i} \alpha_i^{\ell} y_i^{\ell} \right) \leq \sum_{i=1}^m \sum_{\ell=1}^{L_i} g_i(\alpha_i^{\ell}) y_i^{\ell}.$$

Thus, the desired inequality follows. \square

The following example shows that, in general, the inequality given by Proposition 2.4.1 is not an equality.

Example

Consider the convex capacitated assignment problem

$$\text{minimize } 5x_{11} + 10x_{12} + 10x_{21} + 2x_{22}$$

subject to

$$\begin{aligned} 3x_{11} + x_{12} &\leq 2 \\ 3x_{21} + 4x_{22} &\leq 3 \\ x_{11} + x_{21} &= 1 \\ x_{12} + x_{22} &= 1 \\ x_{ij} &\in \{0, 1\} \quad i = 1, 2; j = 1, 2. \end{aligned}$$

The optimal solution vector for the relaxation of the integrality constraints is equal to $x_{11}^* = 4/9$, $x_{21}^* = 5/9$, $x_{12}^* = 2/3$, and $x_{22}^* = 1/3$ with objective value equal to $15 + 1/9$.

With respect to the set partitioning formulation, there only exist two feasible columns per agent due to the capacity constraints. The columns associated with agent 1 are $(0 \ 0)^{\top}$ and $(0 \ 1)^{\top}$, and $(0 \ 0)^{\top}$ and $(1 \ 0)^{\top}$ are the ones of agent 2. It is straightforward to see that the optimal solution of LP(MP) is given by $y_1^1 = 0$, $y_1^2 = 1$, $y_2^1 = 0$ and $y_2^2 = 1$ with objective value equal to 20. We also may observe that the optimal solution of R(CCAP) is not a convex combination of the columns of (MP).

From the proof of Proposition 2.4.1, we can easily see that this result still holds for a more general class of assignment problems where the constraints faced by each

agent are defined by a convex set in \mathbb{R}^{k_i} , for each $i = 1, \dots, m$ (see Freling et al. [52]).

The convex capacitated assignment problem is a (nonlinear) Integer Programming Problem which can be solved to optimality by using, for example, a Branch and Bound algorithm. One of the factors determining the performance of this algorithm is the quality of the lower bounds used to fathom nodes. Proposition 2.4.1 shows that the lower bound given by relaxing the integrality constraints in (MP) is at least as good as the one obtained by relaxing the integrality constraints in the CCAP. Thus, the set partitioning formulation for the convex capacitated assignment problem looks more attractive when choosing a Branch and Bound scheme. There are other reasons to opt for this formulation like the possibility of adding constraints that are difficult to express analytically.

A standard Branch and Bound scheme would require all the columns to be available, but (in the worst case) the number of columns (and thus the number of variables) of (MP) can be exponential in the size of the problem. This makes a standard Branch and Bound scheme quite unattractive for (MP). However, since the number of constraints in (MP) is relatively small with respect to the number of variables, only few variables will have strictly positive value in the optimal solution of LP(MP). Thus, only a very small subset of columns is relevant in the optimization of LP(MP). Basically, this is the philosophy behind Column Generation techniques (see Gilmore and Gomory [61]). Combining a Branch and Bound scheme with a column generation procedure yields a so-called Branch and Price algorithm. In the next we will describe a Branch and Price scheme for (MP).

2.5 A Branch and Price scheme

2.5.1 Introduction

Barnhart et al. [9] have unified the literature on Branch and Price algorithms for large scale Mixed Integer Problems. They focus on branching rules and some computational issues relevant in the implementation of a Branch and Price scheme. We will concentrate mainly on the pricing problem. A similar approach has been followed by Chen and Powell [31] for parallel machine scheduling problems when the objective function is additive in the jobs.

2.5.2 Column generation scheme

Usually, the number of columns associated with each agent will be extremely large, thus prohibiting the construction and solution of LP(MP) as formulated above. However, one may solve LP(MP) using only a subset (say N) of its columns (and refer to the corresponding reduced problem as LP(MP(N))). If it is then possible to check whether this solution is optimal for LP(MP), and to generate an additional column that will improve this solution if it is not, we can solve LP(MP) using a so-called *column generation* approach:

Column generation for LP(MP)

- Step 0.** Construct a set of columns, say $N_0 \subseteq \{(\ell, i) : \ell = 1, \dots, L_i; i = 1, \dots, m\}$, such that $\text{LP}(\text{MP}(N_0))$ has a feasible solution. Set $N = N_0$.
- Step 1.** Solve $\text{LP}(\text{MP}(N))$, yielding $y^*(N)$.
- Step 2.** If $y^*(N)$, extended to a solution of $\text{LP}(\text{MP})$ by setting the remaining variables to zero, is optimal for $\text{LP}(\text{MP})$: STOP.
- Step 3.** Find a column (or a set of columns) so that the new objective value is at least as good as the objective value of $y^*(N)$ and add this column (or set of columns) to N . Go to Step 1.

Steps 2 and 3 verify that the optimal solution of $\text{LP}(\text{MP}(N))$ is also optimal for $\text{LP}(\text{MP})$ or find a new columns to add to $\text{LP}(\text{MP}(N))$ that may improve the current objective value. The information contained in the optimal dual multipliers of the constraints of $\text{LP}(\text{MP}(N))$ is used to perform those steps. In the following we will describe the steps of the algorithm in more detail.

Step 0: Initial columns

The column generation procedure calls for an initial set of columns N_0 to start with. For this purpose, the class of greedy heuristics and the two local exchange procedures proposed in Section 2.3 can be used.

The output of this heuristic is a vector of feasible assignments x^G , which is (at least) a partial solution to the CCAP, and thus yields a set of columns for (MP). As mentioned in the previous section, the optimal dual vector of $\text{LP}(\text{MP}(N_0))$ is required to perform Steps 2 and 3. Thus, when the solution is only a partial one, $\text{LP}(\text{MP}(N_0))$ is infeasible and we cannot start the column generation procedure. Moreover, it could also be the case that (MP) is infeasible. To overcome those two situations we have added a dummy variable $s_j \geq 0$ to the j -th constraint (2.1) with a high cost, for each $j = 1, \dots, n$. This ensures that $\text{LP}(\text{MP}(N_0))$ always has a feasible solution, and infeasibility of this LP-problem is characterized by the positiveness of some of the dummy variables.

Steps 2 and 3: The Pricing Problem

A major issue in the success of the column generation approach is of course the viability of Steps 2 and 3. The usual approach is to consider the dual problem $\text{D}(\text{MP})$ to $\text{LP}(\text{MP})$:

$$\text{maximize } \sum_{j=1}^n u_j - \sum_{i=1}^m \delta_i$$

subject to

$\text{D}(\text{MP})$

$$\sum_{j=1}^n \alpha_{ij}^\ell u_j - \delta_i \leq g_i(\alpha_{i\cdot}^\ell) \quad \ell = 1, \dots, L_i; i = 1, \dots, m$$

$$\begin{array}{lll}
u_j & \text{free} & j = 1, \dots, n \\
\delta_i & \text{free} & i = 1, \dots, m.
\end{array}$$

Note that the optimal dual solution corresponding to $y^*(N)$, say $(u^*(N), \delta^*(N))$, satisfies all dual constraints in $D(MP)$ corresponding to elements $(\ell, i) \in N$. Moreover, if it satisfies *all* dual constraints in $D(MP)$, then $y^*(N)$ (extended with zeroes) is the optimal solution to $LP(MP)$. The challenge is thus to check feasibility of the dual solution $(u^*(N), \delta^*(N))$. This can, for example, be achieved by solving, for each $i = 1, \dots, m$, the following optimization problem

$$\text{minimize } g_i(z) - \sum_{j=1}^n u_j^*(N) z_j + \delta_i^*(N)$$

subject to

$$\begin{array}{ll}
A^i z & \leq b_i \\
z_j & \in \{0, 1\} \quad j = 1, \dots, n
\end{array}$$

thereby finding the minimum slack in all dual constraints. If all these optimization problems yield a nonnegative value, then all dual constraints are satisfied. Otherwise, feasible solutions with positive objective function value correspond to columns that would enter the basis if added to $LP(MP(N))$ (starting from $y^*(N)$).

The success of the column generation procedure depends on the ability to solve this subproblem efficiently, thus, its structure is crucial. For example, Savelsbergh [121] shows that this subproblem turns out to be a Knapsack Problem for the GAP.

2.5.3 Branching rule

If the optimal solution of the LP-relaxation of (MP) is not integer we need to branch to obtain an optimal integer solution. Since the LP-relaxation of (MP) has been solved by column generation, it is unlikely that all columns are present in the final reduced linear programming problem. Thus, by using a Branch and Bound scheme with only the columns in this way generated, we will in the best case end up with only a feasible solution for (MP). This approach thus yields a heuristic for solving the convex capacitated assignment problem.

If we want a certificate of optimality new columns (when needed) should be generated when branching. The choice of the branching rule is crucial since it can destroy the structure of the pricing problem. The straightforward choice would be to branch on the variables y_i^ℓ . Fixing one of those variables to zero is equivalent to prohibiting the generation of that column again. As Savelsbergh [121] pointed out for the GAP, with this branching rule we may need to find not only the optimal solution of the pricing problem but also the second optimal solution for it. Usually we cannot incorporate this additional information into the pricing problem directly, thereby prohibiting an efficient algorithm for the pricing problem. However, the proof of Proposition 2.4.1 shows that each feasible solution y for (MP) has a corresponding feasible solution x for the CCAP. Moreover, it is easy to see that if y is fractional then

x is fractional as well. Thus, we can branch on the fractional variables x_{ij} . We may observe that the subproblems obtained by branching on the x_{ij} variables are again convex capacitated assignment problems. Thus, the column generation procedure in each node of the tree is the same as in the root node.

2.5.4 A special case of CCAP's

In this section we will consider a class of CCAP's for which the pricing problem exhibits an attractive property. We will analyze the class of convex capacitated assignment problems where each agent i faces just one knapsack constraint and the costs g_i are equal to the sum of a linear function in x_i , and a convex penalization of the use of the knapsack of agent i . More precisely, we will choose

$$g_i(z) = \sum_{j=1}^n \nu_{ij} z_j + G_i \left(\sum_{j=1}^n \omega_{ij} z_j \right) \quad \text{for each } z \in \mathbb{R}^n,$$

where $\sum_{j=1}^n \omega_{ij} z_j \leq \Omega_i$ is the knapsack constraint associated with agent i , and $G_i : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function.

We may notice that the GAP is still a member of this class with $G_i = 0$ for each $i = 1, \dots, m$. Some extensions of the GAP are also included. The convex penalty function could be seen as a way of modeling a situation where the resource capacities are not rigid, and where they are allowed to be exceeded at some cost (see Srinivasan and Thompson [126]). Another example could be that a convex penalty is used to model the fact that it is undesirable to plan the use of resources to full capacity, due to possible deviations from the predicted requirements when a solution is implemented.

Since those are still convex capacitated assignment problems, we can use the Branch and Price scheme described above. As already mentioned, the success of this procedure depends on how efficiently we can solve the pricing problem. After rearranging terms and transforming it into a maximization problem, the pricing problem for this class associated with agent i reads

$$\text{maximize } \sum_{j=1}^n (u_j^*(N) - \nu_{ij}) z_j - G_i \left(\sum_{j=1}^n \omega_{ij} z_j \right) - \delta_i^*(N)$$

subject to

$$\begin{aligned} \sum_{j=1}^n \omega_{ij} z_j &\leq \Omega_i \\ z_j &\in \{0, 1\} \quad j = 1, \dots, n. \end{aligned}$$

Without loss of optimality we can leave out the constant term $\delta_i^*(N)$. The feasible region of this problem is described by a knapsack constraint. As in the Knapsack Problem, items which are added to the knapsack yield a profit $u_j^*(N) - \nu_{ij}$. However,

in contrast to the traditional Knapsack Problem, the utilization of the knapsack is penalized by the convex function G_i . We will call this problem the Penalized Knapsack Problem (PKP) and it will be analyzed in the next section.

2.5.5 The Penalized Knapsack Problem

Definition of the problem

Consider a knapsack with a certain capacity and a set of items which make use of this capacity. When adding an item to the knapsack a profit is obtained. However, the total use of the knapsack will be penalized by a convex function. The PKP is the problem of choosing items in such a way that the capacity constraint is not violated when we add those items to the knapsack and the total profit minus the penalization on the use of the knapsack is maximal.

Let n denote the number of items. The required space of item j is given by $\omega_j \geq 0$, and the profit associated with adding item j to the knapsack is equal to $p_j \geq 0$. Let Ω be the capacity of the knapsack, and let $G(u)$ denote the penalization of using u units of capacity of the knapsack, where G is a convex function. The PKP can then be formulated as follows:

$$\text{maximize } \sum_{j=1}^n p_j z_j - G\left(\sum_{j=1}^n \omega_j z_j\right)$$

subject to

$$\begin{aligned} \sum_{j=1}^n \omega_j z_j &\leq \Omega \\ z_j &\in \{0, 1\} \quad j = 1, \dots, n. \end{aligned}$$

Since all items with zero space requirement will definitely be added to the knapsack, we can without loss of generality assume that $\omega_j > 0$ for each $j = 1, \dots, n$. However, contrary to the ordinary knapsack problem, items with $p_j = 0$ cannot a priori be excluded from the knapsack, since the penalty function is not required to be nondecreasing, and thus adding such an item to the knapsack can be profitable. If the penalization on the use of the knapsack is nonpositive, i.e. $G(u) \leq 0$ for each $u \in [0, \Omega]$, we know that the optimal solution of the problem is maximal in the sense that no additional items can be added to the knapsack without violating the capacity constraint, see Martello and Toth [89]. However, in the general case, it may occur that the profit associated with adding an item to the knapsack is not enough to compensate for the penalization of the capacity used to add this item to the knapsack. The same follows for the relaxation of the PKP, say R(PKP), where the integer constraints are relaxed.

Consider the case where some items have already been added to the knapsack. Let u be the used capacity by those items. We will say that item j not yet in the knapsack is a *feasible* item for the knapsack if

$$u + \omega_j \leq \Omega,$$

and that it is *profitable* if

$$p_j - G(u + \omega_j) = \max_{\gamma \in [0,1]} \{p_j \gamma - G(u + \omega_j \gamma)\}.$$

Example

Consider the following example of the PKP where there is only one item to be added to the knapsack ($n = 1$) with profit $p_1 = 1$ and required space $\omega_1 = 4$. Moreover, let the capacity of the knapsack be equal to $\Omega = 5$ and the penalization equal to $G(u) = 2u^2$. This particular instance of the PKP reads

$$\text{maximize } z_1 - 2z_1^2$$

subject to

$$\begin{aligned} 4z_1 &\leq 5 \\ z_1 &\in \{0, 1\}. \end{aligned}$$

The item is feasible since the required space (4) is below the capacity (5). The objective value of not adding the item to the knapsack ($z_1 = 0$) is equal to 0, and the cost of adding it to the knapsack completely is equal to -1 . Thus, the item is not profitable and the optimal solution of the PKP is equal to 0. Figure 2.1 plots the value of its relaxation $R(\text{PKP})$. We may observe that the maximum of this function is attained at $z_1^* = 0.25$ even though, as we have seen before, the item can be feasibly added to the knapsack.

The relaxation

One of the properties of the PKP is that the optimal solution of its relaxation $R(\text{PKP})$ has the same structure as the optimal solution of the LP-relaxation of the standard Knapsack Problem (see Martello and Toth [89]), and can be solved explicitly as well.

Assume that the items are ordered according to non-increasing ratio p_j/ω_j . Assume that item 1 until item $\ell - 1$ can be feasibly added to the knapsack. Let $P^\ell(\gamma)$, where $\gamma \in [0, 1]$, be the objective value of $R(\text{PKP})$ associated with the solution $z_j = 1$ for each $j = 1, \dots, \ell - 1$, $z_j = 0$ for each $j = \ell + 1, \dots, n$, and $z_\ell = \gamma$ regardless feasibility. The next lemma shows the behaviour of this function.

Lemma 2.5.1 $P^\ell(\cdot)$ is a concave function.

Proof: The result follows by observing that

$$P^\ell(\gamma) = \sum_{j=1}^{\ell-1} p_j + p_\ell \gamma - G\left(\sum_{j=1}^{\ell-1} \omega_j + \omega_\ell \gamma\right)$$

which a concave function in γ . □

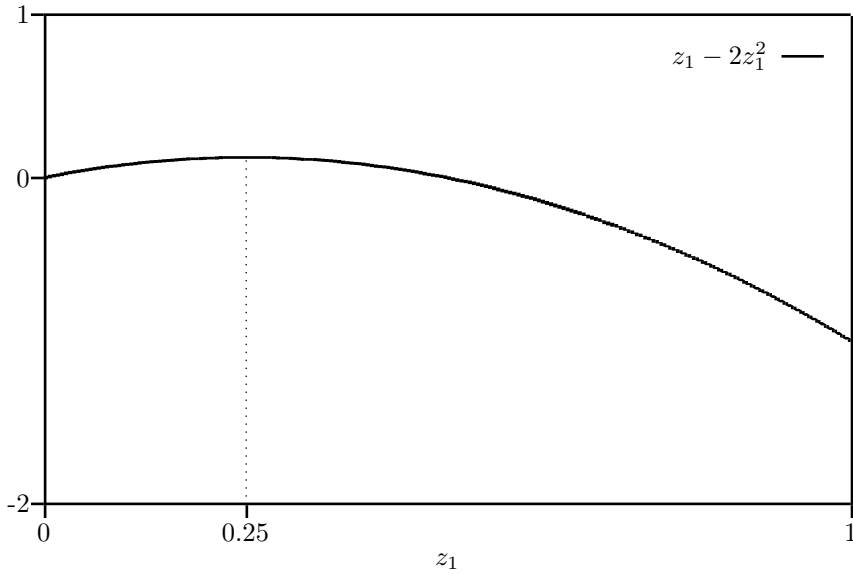


Figure 2.1: Value of the relaxation R(PKP)

Given that item 1 until item $\ell - 1$ have been added to the knapsack, item ℓ is profitable if the maximum of the function $P^\ell(\cdot)$ is reached at $\gamma = 1$. This can be characterized by the condition $(P^\ell)'_-(1) \geq 0$, where $(P^\ell)'_-(\gamma)$ denotes the left derivative of the function P^ℓ in γ . Define items k_1 and k_2 as

$$\begin{aligned} k_1 &= \min\{\ell = 1, \dots, n : \sum_{j=1}^{\ell} \omega_j > \Omega\} \\ k_2 &= \min\{\ell = 1, \dots, n : (P^\ell)'_-(1) < 0\}. \end{aligned}$$

By definition, k_1 is the first item which cannot be added completely to the knapsack due to the capacity constraint, and item k_2 is the first item which will not be added completely to the knapsack due to the penalization of the capacity utilization. Now define item k as

$$k = \min\{k_1, k_2\},$$

i.e., the first item which should not be completely added to the knapsack due to either the capacity constraint or because it is not profitable. In the next proposition we show that the optimal solution for R(PKP) just adds to the knapsack items $1, \dots, k - 1$ and the feasible and profitable fraction γ^* of item k , i.e.,

$$\gamma^* = \min\{\gamma_1^*, \gamma_2^*\}$$

where

$$\gamma_1^* = \frac{\Omega - \sum_{j=1}^{k-1} \omega_j}{\omega_k}$$

and γ_2^* is the maximizer of the function $P^k(\cdot)$.

Proposition 2.5.2 *The vector $\bar{z} \in \mathbb{R}^n$ defined by*

$$\bar{z}_j = \begin{cases} 1 & \text{if } j < k \\ \gamma^* & \text{if } j = k \\ 0 & \text{if } j > k \end{cases}$$

is an optimal solution for R(PKP).

Proof: Let z be a feasible solution for R(PKP). The idea is to show that there exists a feasible solution \hat{z} at least as good as z so that $\hat{z}_j = 1$ for each $j = 1, \dots, k-1$ and $\hat{z}_j = 0$ for each $j = k+1, \dots, n$. By the definition of item k and fraction γ^* , \bar{z} is at least as good as \hat{z} . Thus, the desired result follows.

Suppose that there exists an item $r = 1, \dots, k-1$ so that $z_r < 1$. If $z_q = 0$ for each $q = k, \dots, n$ we can construct a better solution by increasing z_r to 1 because the first $k-1$ items are feasible and profitable. Thus, assume that there exists $q = k, \dots, n$ so that $z_q > 0$. By increasing z_r by $\varepsilon > 0$ and decreasing z_q by $\varepsilon \omega_r / \omega_q$ the used capacity remains unchanged which implies that the penalization remains the same. Moreover, the profit associated with the new solution is at least as good as the profit in z since

$$p_r \varepsilon - p_q \varepsilon \omega_r / \omega_q = \varepsilon \omega_r (p_r / \omega_r - p_q / \omega_q) \geq 0$$

because $r < q$. Hence, we can assume that $z_j = 1$ for each $j = 1, \dots, k-1$.

Now we will prove that $z_j = 0$ for each $j = k+1, \dots, n$. Suppose that there exists an item $r = k+1, \dots, n$ so that $z_r > 0$. Then, $z_k < \gamma_1^*$ since $z_j = 1$ for each $j = 1, \dots, k-1$. In this case, by increasing z_k by $\varepsilon > 0$ and decreasing z_r by $\varepsilon \omega_r / \omega_q$ it follows in a similar way as above that the new solution is at least as good as z . \square

Lemma 2.5.1 and Proposition 2.5.2 suggest a procedure to solve R(PKP) explicitly. We will denote the optimal solution for R(PKP) by z^R and a feasible solution for the PKP by z^{IP} . We will add items to the knapsack while there is enough space and the objective function does not decrease, i.e., $P^\ell(1) \geq P^\ell(0)$. Let r be the last item added to the knapsack. If we stop due to infeasibility, then the critical item is $k = r+1$. Otherwise, the objective function decreases if item $r+1$ is completely added to the knapsack. Then, there are two possible cases. In the first case, the function P^r is an increasing function, thus the item $k = r+1$ is the first item which is not profitable. Otherwise, the maximum of the function P^r is attained at $\gamma \in (0, 1)$, so this is the critical item, i.e. $k = r$. However, we only realize that when we try to add item $r+1$. More precisely, if $(P^r)'(1) \geq 0$ then $k = r+1$, otherwise $k = r$. Finally, it remains to evaluate the optimal fraction γ_k^* which can be found efficiently since it is the maximizer of a concave function (see Hiriart-Urruty and Lemaréchal [70]). We may observe that as a by-product we obtain a feasible solution z^{IP} for the PKP. We can set $z_j^{\text{IP}} = 1$ for each $j = 1, \dots, r$ and $z_j^{\text{IP}} = 0$ otherwise.

Recall that the items have been renumbered so that if $j < k$ then $\frac{p_j}{\omega_j} \geq \frac{p_k}{\omega_k}$.

Solving R(PKP)

Step 0. Set $J = \{1, \dots, n\}$. Set $z_j^R = 0$ and $z_j^{IP} = 0$ for each $j = 1, \dots, n$.

Step 1. Set $\hat{j} = \arg \min\{j \in J\}$ and $J = J \setminus \{\hat{j}\}$. If \hat{j} is not feasible then set $k = \hat{j}$ and go to Step 3.

Step 2. If $P^{\hat{j}}(1) \geq P^{\hat{j}}(0)$, set

$$\begin{aligned} z_{\hat{j}}^{IP} &= 1 \\ z_{\hat{j}}^R &= 1 \end{aligned}$$

and go to Step 1. Else, if $(P^{\hat{j}-1})'_-(1) \geq 0$ set $k = \hat{j}$, else set $k = \hat{j} - 1$.

Step 3. Set

$$z_{\hat{j}}^R = \min \left\{ \arg \max_{\gamma \in [0,1]} P^k(\gamma), \frac{\Omega - \sum_{j=1}^{k-1} \omega_j}{\omega_k} \right\},$$

and STOP.

Step 2 is illustrated by Figures 2.2 and 2.3. In the first case (see Figure 2.2), item $k-1$ was added completely since P^{k-1} is a strictly increasing function. However, the objective function drops from 8 to 7.81 by adding item k to the knapsack. Thus, this is the first item which is not profitable. However, in the second case (see Figure 2.3) the objective function increases from 5 to 6 by adding item k to the knapsack. Nevertheless, the maximum of the objective function is attained at $\gamma_k = 2/3$, so this is the critical item. However, we only realize that after we try to add item $k+1$.

2.6 Summary

In the rest of this thesis we will frequently refer to the results developed for the CCAP in this chapter. Therefore, we will briefly summarize them. These results concern with solving the CCAP and the generation of suitable problem instances for this problem to test solution procedures. We have defined a general stochastic model for the CCAP and have found an implicit tight condition to ensure asymptotic feasibility in the probabilistic sense of the problem instances generated by it. We have defined a class of greedy heuristics to obtain feasible solutions for the CCAP. The solution obtained by these greedy heuristics has been improved by two local exchange procedures. Finally, we have proposed a Branch and Price scheme to solve the set partitioning formulation given for the CCAP in Section 2.4. Two critical factors for the Branch and Price scheme have been analyzed, namely the structure of the pricing problem and branching rules which are compatible with the pricing problem.

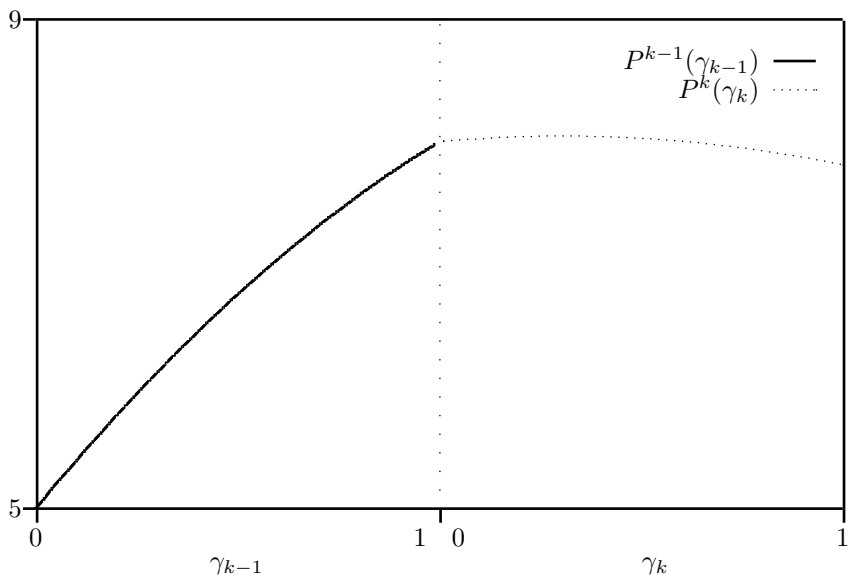


Figure 2.2: When $k = r + 1$

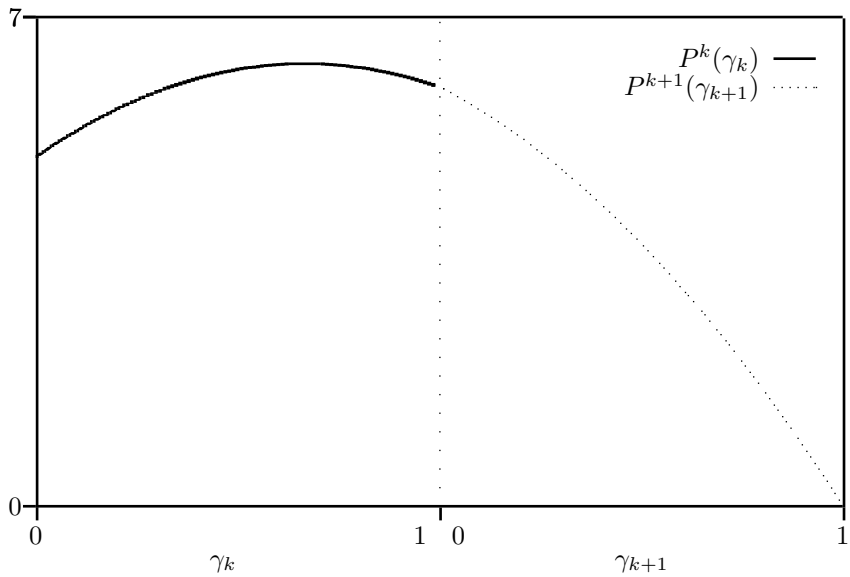


Figure 2.3: When $k = r$

Part II

Supply Chain Optimization
in
a static environment

Chapter 3

The Generalized Assignment Problem

3.1 Introduction

The competition in the marketplace and the shortening of the life-cycle of the products are examples of reasons that force companies to continuously enhance the performance of their logistics distribution networks. New opportunities for improving the performance of the logistics distribution network may appear after introducing a new product in the market, the merger of several companies, the reallocation of the demand, etc. The (re)design of the logistics distribution network of a company involves (re)considering the product flows from the producers to the customers, possibly through distribution centers (warehouses), and the handling of the products at each of the levels of this logistics distribution network. Examples of decisions related to the handling are the selection of an inventory policy at the warehouses, or the transportation modes used at the first transportation level of the logistics distribution network from the plants to the warehouses as well as the second level from the warehouses to customers.

Establishing an appropriate measure representing the efficiency/effectiveness of the logistics distribution network is one of the main tasks when evaluating the performance of this network, see Beamon [11]. As mentioned before, a commonly used measure is an estimate of the total costs which includes production, handling, inventory holding, and transportation costs. This estimate is quantified by a model which selects the optimal location and size of production and inventory, and determines the allocation of customers to facilities (plants or warehouses), subject to a number of constraints regarding capacities at the facilities, assignments, perishability, etc. Models associated with the estimate of the total costs for a given design of the logistics distribution network contain an assignment structure due to the allocation of the customers to facilities. Moreover, those assignments are constrained by the capacities at the facilities. Consequently, a good understanding of the capacitated assignment models will help us when dealing with more complex structures.

The *Generalized Assignment Problem (GAP)* is the simplest version of a capacitated assignment model. The GAP is suitable when evaluating the production, handling, and transportation costs of a logistics distribution network where production and storage take place at the same location. The GAP is, by nature, a static model so inventory decisions are not explicitly modeled. In Chapter 6, we will show that some of the dynamic models analyzed in this thesis can be modeled as GAP's with nonlinear objective function. The relevance of the GAP in both static and dynamic models has induced us to devote Part II to the analysis of this problem.

The outline of this chapter is as follows. In Section 3.2 we introduce the standard formulation of the GAP. In Section 3.3 we summarize the literature devoted to the GAP. Section 3.4 presents several extensions of the GAP. Finally, Section 3.5 shows an attractive property of the Linear Programming Relaxation (hereafter LP-relaxation) of the GAP. Some of the results in this chapter can be found in Romeijn and Romero Morales [115].

3.2 The model

In the GAP there are tasks which need to be processed and agents which can process them. A single resource available to the agents is consumed when processing the tasks. Each agent has a given capacity for the resource, and the requirement, or resource consumption, of each task may depend on the agent processing it. The GAP is then the problem of assigning each task to exactly one agent, so that the total cost of processing the tasks is minimized and each agent does not exceed its capacity. The problem can be formulated as an integer linear programming problem as follows:

$$\text{minimize } \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to

$$\sum_{j=1}^n a_{ij} x_{ij} \leq b_i \quad i = 1, \dots, m \quad (3.1)$$

$$\sum_{i=1}^m x_{ij} = 1 \quad j = 1, \dots, n \quad (3.2)$$

$$x_{ij} \in \{0, 1\} \quad i = 1, \dots, m; j = 1, \dots, n$$

where the cost coefficients c_{ij} , the requirement coefficients a_{ij} , and the capacity parameters b_i are all nonnegative scalars. Constraints (3.2) are known in the literature as semi-assignment constraints. Some authors have used a maximization formulation of the problem, see for example Martello and Toth [88], Fisher, Jaikumar and Van Wassenhove [48], and Savelsbergh [121]. As mentioned in Chapter 2, the GAP is a member of the CCAP where the cost function g_i associated with agent i is linear in x_i . and just one capacity constraint is faced by each agent, i.e., $k_i = 1$ for each $i = 1, \dots, m$.

The GAP was defined by Ross and Soland [118], and is inspired by real-life problems such as assigning jobs to computer networks (see Balachandran [5]) and fixed charge plant location where customer requirements must be satisfied by a single plant (see Geoffrion and Graves [58]). Other applications that have been studied are the p -median location problem (see Ross and Soland [119]), the maximal covering location problem (see Klasterin [76]), routing problems (see Fisher and Jaikumar [47]), R & D planning problems (see Zimokha and Rubinshtein [140]), and the loading problem in flexible manufacturing systems (see Kuhn [82]). Various approaches can be found to solve this problem, most of which were summarized by Cattrysse and Van Wassenhove [26] and Osman [101].

The Single Sourcing Problem (hereafter SSP) is a particular case of the GAP where the requirements are agent-independent, i.e. $a_{ij} = d_j$ for each $i = 1, \dots, m$. This problem was in fact introduced before the GAP by De Maio and Roveda [34]. They interpret the SSP as a special transportation problem where each demand point must be supplied by exactly one source. Allocating the items necessary for production and maintenance operations in a set of warehouses in order to minimize delivery costs originated the SSP. Srinivasan and Thompson [126] propose agent-dependent requirements as an extension of the SSP, i.e., what is now known as the GAP. In Part III we study another extension of the SSP, the Multi-Period Single-Sourcing Problem, where the demand and the capacities are time-varying and capacity can be transferred to future periods.

Due to its interest, this problem has been studied from an algorithmic point of view extensively. Different exact algorithms and heuristics have been proposed in the literature. Nevertheless, all approaches suffer from the \mathcal{NP} -Hardness of the GAP (see Fisher, Jaikumar and Van Wassenhove [48]). Moreover, the decision problem associated with the feasibility of the GAP is an \mathcal{NP} -Complete problem (see Martello and Toth [89]). (See Garey and Johnson [54] for a definition of \mathcal{NP} -Hardness and \mathcal{NP} -Completeness.) Therefore, even to test whether a problem instance has at least one feasible solution is computationally hard. In their proofs, Fisher, Jaikumar and Van Wassenhove [48] and Martello and Toth [89] use problem instances of the GAP with agent-independent requirements. Hence, the complexity results also hold for the SSP.

3.3 Existing literature and solution methods

In this section we summarize the research devoted to the GAP. We first concentrate on algorithmic developments. In this respect, we present bounding techniques which has been incorporated in branch and bound schemes to solve to optimality the problem. We then describe heuristic and meta-heuristic approaches for the GAP. Finally, we present other studies of the GAP.

Different bounds for the GAP have been proposed to be embedded in a branch and bound scheme. Ross and Soland [118] relax the capacity constraints (3.1) to $a_{ij}x_{ij} \leq b_i$. This turns into a simple cost minimization problem where each task is assigned to its cheapest (feasible) agent in the optimal solution. In general, this solution violates the capacity constraints of some of the agents. Therefore, some

tasks must be reassigned yielding a penalty of the objective value. The bound is improved by adding the minimal penalty incurred to avoid this violation. Martello and Toth [88] show that the algorithm of Ross and Soland is not fast enough when the capacity constraints are tight. Instead, they propose to calculate bounds by removing the semi-assignment constraints. The relaxed problem decomposes into m knapsack problems. Again, the corresponding bound is improved by adding a lower bound on the penalty to be paid to satisfy the violated semi-assignment constraints. Fisher, Jaikumar and Van Wassenhove [48] obtain bounds using Lagrangean relaxation in the semi-assignment constraints. A multiplier adjustment method (see Fisher [45]) is used to find good multipliers. Guignard and Rosenwein [66] observe that the largest test-problems reported in the literature contained at most 100 variables. They propose some enhancements and additions to the approach of Fisher, Jaikumar and Van Wassenhove [48] to be able to solve larger problems. First, they enlarge the set of possible directions used by the multiplier adjustment method. Second, if the obtained solution violates $\sum_{i=1}^m \sum_{j=1}^n x_{ij} = n$, then the corresponding surrogate constraint ($\sum_{i=1}^m \sum_{j=1}^n x_{ij} \leq n$ or $\sum_{i=1}^m \sum_{j=1}^n x_{ij} \geq n$) is added to improve the bound given by the heuristic. They were able to solve problems with 500 variables. Karabakal, Bean and Lohmann [74] argue that the multiplier adjustment methods proposed by Fisher, Jaikumar and Van Wassenhove [48] and Guignard and Rosenwein [66] move along the first descent direction (for a maximization formulation of the GAP) found with nonzero step size. They propose to use the steepest descent direction. Unfortunately, no comparisons are shown with the two previous methods. Linear programming bounds have been used by Jörnsten and Värbrand [72], improved by valid inequalities from the knapsack constraints. They also consider new branching rules. Numerical results are only shown for problem instances of size $m = 4$ and $n = 25$. Cattrysse, Degraeve and Tistaert [23] also strength the linear relaxation of the GAP by valid inequalities. Savelsbergh [121] proposes a branch and price algorithm for the set partitioning formulation of the GAP (see Section 2.5).

Due to the hardness of the GAP, a significant number of heuristic procedures have been proposed. First, we describe those ones based on the LP-relaxation of the GAP. Benders and Van Nunen [14] prove that the number of infeasible tasks, i.e. the ones assigned to more than one agent, in the optimal solution for the LP-relaxation is at most the number of agents used to full capacity. They propose a heuristic that assigns the infeasible tasks of the optimal solution for the LP-relaxation of the GAP. Cattrysse [24] proposes to fix the feasible tasks of the LP-relaxation of the GAP and solve the reduced problem by a branch and bound technique. He observes that adding cuts to the LP-relaxation increases the number of fractional variables, so that less tasks are fixed. Numerical experiments show that this increases the success of his primal heuristic. Trick [131] uses another property of the LP-relaxation of the GAP to propose his LR-heuristic. He defines the variable associated with the assignment of task j to agent i useless if the requirement of task j on agent i is larger than the capacity of agent i , which means that without loss of optimality useless variables can be fixed to zero. The basic idea is to solve the LP-relaxation of the GAP and fix all the feasible tasks. We then obtain a new GAP with the same number of agents which has at most m tasks (see Benders and Van Nunen [14]). Moreover, useless variables

are fixed to zero. This procedure is successively repeated.

A considerable number of the heuristic approaches for the GAP are based on Lagrangean relaxations. Chalmet and Gelders [27] use subgradient optimization for the two possible Lagrangean relaxations. They notice that the constraint matrix when relaxing the capacity constraints is totally unimodular. Thus, this bound coincides with the optimal value of the LP-relaxation of the GAP, see Geoffrion [57]. Nevertheless, they claim that the former one can be calculated more efficiently for large problem instances. By relaxing the semi-assignment constraints, better bounds are expected since the unimodularity property does not hold. Klastorin [75] uses a subgradient method for the relaxation of the capacity constraints. A branch and bound scheme was implemented to search in the neighborhood of the current solution. Jörnsten and Näsberg [71] apply a Lagrangean decomposition approach (see Guignard and Kim [65]). This approach enables to combine the two structures obtained by Lagrangean relaxation. Moreover, this bound is at least as good as these two Lagrangean relaxations. A subgradient method is used to find bounds, and heuristic procedures try to find primal solutions. There is no description of the way the test-problem instances were generated. Barcia and Jörnsten [8] use the bound improving technique (see Barcia [7]) to tighten the bound obtained by Lagrangean decomposition. Lorena and Narciso [85] propose a subgradient method for the Lagrangean relaxation of the capacity constraints and the surrogate relaxation of those ones. Primal solutions are searched for by a greedy heuristic from the class of Martello and Toth [88] and a constructive heuristic. In a later work, Narciso and Lorena [97] propose a Lagrangean/surrogate relaxation. The numerical results are averaged output for all the problem instances tested.

Cattrysse, Salomon and Van Wassenhove [25] propose a heuristic based on the set partitioning formulation of the GAP. They solve its LP-relaxation by a multiplier adjustment method combined with a subgradient method. They look for primal solutions by reduction techniques.

Martello and Toth [88] propose one of the most widely used greedy heuristics for the GAP (see Section 2.3.1). They also add a local search phase where they try to improve the objective value of the current solution. Wilson [137] uses the solution where each task is assigned to its cheapest agent as the starting point for an exchange procedure where the violation of the capacity constraints is decreased in each step.

Meta-heuristics have also been proposed for the GAP. Cattrysse [24] implements a simulating annealing concluding that it is only competitive for small problem sizes. Racer and Amini [105] describe a variable depth search heuristic (where the main idea is to adaptively change the size of the neighborhood). They compare their results with the heuristic from Martello and Toth [88] on five classes of problem. At the expense of high computation times, the variable depth search heuristic finds better feasible solutions than the greedy heuristic for one of the problem classes. In order to decrease computation times, Amini and Racer [2] describe a hybrid heuristic where initial solutions are generated with the heuristic from Martello and Toth [88] and refined with a variable depth search heuristic. Osman [101] proposes a simulating annealing and a tabu search. Chu and Beasley [32] and Wilson [136] propose genetic algorithms for the GAP. In the first one, a family of potential solutions is generated, and steps are

made to improve feasibility and optimality. On the contrary, good starting solutions in term of objective value are assumed in the second one. Ramalhinho Lourenço and Serra [106] propose two meta-heuristic approaches for the GAP. The first one is a greedy adaptive search heuristic (see Feo and Resende [42] for a general description of such GRASP heuristics), and the second one is a MAX-MIN ant system (see Stützle and Hoos [127]). Both of them are combined with a local search and a tabu search schemes. Yagiura, Yamaguchi and Ibaraki [139] notice that searching only in the feasible region may be too restrictive. Therefore, they propose a variable depth search heuristic where it is allowed to move to infeasible solutions of the problem. Yagiura, Ibaraki and Glover [138] propose an ejection chain approach combined with a tabu search.

Shmoys and Tardos [123] propose a polynomial-time algorithm that, given $C \in \mathbb{R}$, either proves that there is no feasible solution for the GAP with cost C or find a feasible assignment of cost at most C with a consumption of the resource at agent i of at most $2b_i$ for all i . We can also mention an aggregation/disaggregation technique for large scale GAP's proposed by Hallefjord, Jörnsten and Värbrand [69].

Apart from algorithms solving the GAP, there are papers devoted to other aspects. In this respect, Gottlieb and Rao [63, 64] perform a polyhedral study of the GAP. It is straightforward to see that any valid inequality for the Knapsack Problem is also valid for the GAP; they also prove that each facet of the Knapsack Problem is also a facet for the GAP. They found other valid inequalities based upon more than one knapsack constraint. Amini and Racer [1] present an experimental design for computational comparison of the greedy heuristic of Martello and Toth [88] and a variable depth search heuristic which is apparently the same one as the authors have proposed in Racer and Amini [105]. Stochastic models for the GAP have been proposed by Dyer and Frieze [38] and Romeijn and Piersma [110]. In the latter paper a probabilistic analysis of the optimal solution of the GAP under this stochastic model was performed, studying the asymptotic behaviour of the optimal solution value as the number of tasks n goes to infinity. Furthermore, a tight condition on the stochastic model under which the GAP is feasible with probability one when the number of tasks goes to infinity is derived.

3.4 Extensions

As shown in Section 3.2, the GAP appears as part of many real life problems. Some new issues have arisen when modeling those situations. This has yielded to many extensions of the GAP. In the following we summarize some of them.

Srinivasan and Thompson [126] propose a branch and bound procedure for the SSP. They mention a practical extension of this model where the capacity of the agents can be increased at a certain cost. Neebe and Rao [98] consider a fixed-charge version of the SSP where each agent processing at least one task incurs a fixed cost. This can be used for example to model setup costs for the agents. They formulate it as a set partitioning problem and solve it by a column generation scheme. These two generalizations of the SSP also hold for the GAP. As shown by Ross and Soland [119], some location problems can be modeled as a GAP. When considering the location

of emergency service facilities, a min-max objective function is more adequate than the classical min-sum. Mazzola and Neebe [93] consider two min-max formulations of the GAP. The first one is called the Task Bottleneck Assignment Problem (Task BGAP) where the objective function is equal to

$$\max_{i=1,\dots,m; j=1,\dots,n} c_{ij}x_{ij}.$$

The Task BGAP can model the location of emergency service facilities when the response time must be minimized. The second version is the Agent Bottleneck Assignment Problem (Agent BGAP) where the objective function is equal to

$$\max_{i=1,\dots,m} \sum_{j=1}^n c_{ij}x_{ij}.$$

This problem arises in the machine loading problem. There we must minimize the makespan, which implies an assignment of tasks to agents where the maximum consumption of the agents must be minimized. Martello and Toth [90] propose different relaxations for the Task BGAP, and a branch and bound procedure. Mazzola [91] considers nonlinear capacity interactions in the GAP. He mentions that this problem can be found in hierarchical production planning problems where product families must be assigned to production facilities and a changeover is translated into nonlinear capacity interactions between the product families assigned to the same facility. By linearizing the functions defining the capacity constraints, he defines a relaxation of the nonlinear GAP which is a GAP. As mentioned in the introduction, the GAP assumes that there is just one resource available to the agents. Gavish and Pirkul [55] propose the Multi-Resource Generalized Assignment Problem (hereafter MRGAP) where tasks consume more than one resource when being processed by the agents. They present different relaxations and heuristic procedures which are incorporated in a branch and bound scheme. Campbell and Langevin [21] show an application of the MRGAP in the assignment of snow removal sectors to snow disposal sites in the city of Montreal. Blocq [16] shows another application in the distribution of gasoline products. There, an oil company is interested in minimizing the costs when delivering their oil products (super, unleaded petrol, etc.) from the depots to the different petrol stations under some restrictions. For each depot and type of product the total amount delivered has to be in a certain range. The same has to hold for the aggregation of all products, and also for some subsets of products. This problem can be modeled as a MRGAP with lower bounds on the consumption of the resources. He mentions that the company makes sometimes agreements with other oil companies to share their depots to a certain extent. To be able to deal with those restrictions, Blocq et al. [17] extend the MRGAP considering constraints on the consumption of each resource for each subset of agents. The multi-level generalized assignment problem was first describe by Glover, Hultz and Klingman [62]. In this case, agents can process tasks with different levels of efficiency, and tasks must be assigned to agents at a specific level of efficiency. Laguna et al. [83] propose a tabu search algorithm using ejection chains to define neighborhood structures for movements.

Other extensions consider a continuous-time version of the GAP incorporating the order in which the tasks are processed by the agents. In those models tasks can be split over different agents. Kogan, Shtub and Levit [78] propose this extension for the GAP and Shtub and Kogan [124] for the MRGAP.

3.5 The LP-relaxation

The LP-relaxation of the GAP has been studied extensively in the literature. As mentioned above, Benders and Van Nunen [14] show that the number of infeasible tasks, i.e. the ones assigned to more than one agent, in the optimal solution of the LP-relaxation is at most the number of agents used to full capacity. Dyer and Frieze [38] also show that the number of fractional variables is at most equal to two times the number of agents. We prove a result which characterizes infeasible tasks in the optimal solution of the LP-relaxation of the GAP. This result is used in Chapter 5 to prove asymptotic optimality of two greedy heuristics for the GAP.

The linear programming relaxation (LPR) of the GAP reads

$$\text{minimize } \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to

(LPR)

$$\begin{aligned} \sum_{j=1}^n a_{ij} x_{ij} &\leq b_i & i = 1, \dots, m \\ \sum_{i=1}^m x_{ij} &= 1 & j = 1, \dots, n \\ x_{ij} &\geq 0 & i = 1, \dots, m; j = 1, \dots, n \end{aligned} \quad (3.3)$$

Throughout this section we will assume that the feasible region of (LPR) is nonempty.

If the optimal solution for (LPR), say x^{LPR} , does not contain any fractional variable, then this clearly is the optimal solution for the GAP as well. In general, however, this is not the case. We call a task j a *non-split task* of (LPR) if there exists an index i such that $x_{ij}^{\text{LPR}} = 1$. The remaining tasks, called *split tasks*, are assigned to more than one agent. In the following we show a relationship between the number of split tasks, the number of split assignments, and the number of agents used to full capacity. Let F be the set of fractional variables in the optimal solution of (LPR), x^{LPR} , B the set of split tasks in x^{LPR} , and M the set of agents used to full capacity in x^{LPR} , i.e.

$$\begin{aligned} F &= \{(i, j) : 0 < x_{ij}^{\text{LPR}} < 1\} \\ B &= \{j : \exists (i, j) \in F\} \\ M &= \left\{ i : \sum_{j=1}^n a_{ij} x_{ij}^{\text{LPR}} = b_i \right\}. \end{aligned}$$

Lemma 3.5.1 *If (LPR) is non-degenerate, then for a basic optimal solution x^{LPR} of (LPR) we have*

$$|F| = |M| + |B|.$$

Proof: Denote the surplus variables corresponding to the capacity constraints (3.3) by s_i ($i = 1, \dots, m$). Then, (LPR) can be reformulated as

$$\text{minimize } \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

subject to

$$\begin{aligned} \sum_{j=1}^n a_{ij} x_{ij} + s_i &= b_i & i = 1, \dots, m \\ \sum_{i=1}^m x_{ij} &= 1 & j = 1, \dots, n \\ x_{ij} &\geq 0 & i = 1, \dots, m; j = 1, \dots, n \\ s_i &\geq 0 & i = 1, \dots, m. \end{aligned}$$

With some abuse of notation, we will still call this model (LPR). Let $(x^{\text{LPR}}, s^{\text{LPR}})$ be the optimal solution of (LPR). Then, the set M defined above is equal to the set of indices i where $s_i^{\text{LPR}} = 0$.

Under non-degeneracy, the number of nonzero variables in $(x^{\text{LPR}}, s^{\text{LPR}})$ is equal to $n + m$, the number of constraints in (LPR). The number of nonzero assignment variables is equal to $(n - |B|) + |F|$, where the first term corresponds to the variables satisfying $x_{ij}^{\text{LPR}} = 1$, and the second term to the fractional assignment variables. Furthermore, there are $m - |M|$ nonzero surplus variables. Thus we obtain

$$n + m = (n - |B|) + |F| + (m - |M|)$$

which implies the desired result. \square

Some properties are derived for the dual programming problem corresponding to (LPR). Let (D) denote the dual problem of (LPR). Problem (D) can be formulated as

$$\text{maximize } \sum_{j=1}^n v_j - \sum_{i=1}^m b_i \lambda_i$$

subject to

$$\begin{aligned} v_j &\leq c_{ij} + a_{ij} \lambda_i & i = 1, \dots, m; j = 1, \dots, n \\ \lambda_i &\geq 0 & i = 1, \dots, m \\ v_j &\text{ free} & j = 1, \dots, n. \end{aligned} \tag{D}$$

Observe that the capacity constraints have been reformulated as \geq -constraints, so that their dual multipliers are nonnegative. Under non-degeneracy of (LPR), non-split tasks can be distinguished from split tasks using the dual optimal solution, as the following proposition shows.

Proposition 3.5.2 *Suppose that (LPR) is non-degenerate. Let x^{LPR} be a basic optimal solution for (LPR) and let (λ^*, v^*) be the corresponding optimal solution for (D). Then,*

(i) *For each $j \notin B$, $x_{ij}^{\text{LPR}} = 1$ if and only if*

$$c_{ij} + \lambda_i^* a_{ij} = \min_{\ell=1, \dots, m} (c_{\ell j} + \lambda_\ell^* a_{\ell j}),$$

and

$$c_{ij} + \lambda_i^* a_{ij} < \min_{\ell=1, \dots, m; \ell \neq i} (c_{\ell j} + \lambda_\ell^* a_{\ell j}).$$

(ii) *For each $j \in B$, there exists i so that*

$$c_{ij} + \lambda_i^* a_{ij} = \min_{\ell=1, \dots, m; \ell \neq i} (c_{\ell j} + \lambda_\ell^* a_{\ell j}).$$

Proof: Observe that $v_j^* = \min_{\ell=1, \dots, m} (c_{\ell j} + \lambda_\ell^* a_{\ell j}) \geq 0$ for each $j = 1, \dots, n$. Thus, without loss of optimality, we can add to (D) nonnegativity constraints for the variables v_j . By adding surplus variables s_{ij} to the constraints in (D), we obtain the following alternative formulation of the dual problem:

$$\text{maximize } \sum_{j=1}^n v_j - \sum_{i=1}^m b_i \lambda_i$$

subject to

$$\begin{aligned} v_j + s_{ij} &= c_{ij} + a_{ij} \lambda_i & i = 1, \dots, m; j = 1, \dots, n \\ \lambda_i &\geq 0 & i = 1, \dots, m \\ v_j &\geq 0 & j = 1, \dots, m \\ s_{ij} &\geq 0 & i = 1, \dots, m; j = 1, \dots, n. \end{aligned}$$

Let (λ^*, v^*, s^*) be the optimal solution of (D). For each $j \in B$ there exist at least two variables x_{ij}^{LPR} that are strictly positive. Hence, by the complementary slackness conditions, there exist at least two variables s_{ij}^* that are equal to zero. This proves Claim (ii).

To prove Claim (i), it is enough to show that for each $j \notin B$ there exists exactly one variable s_{ij}^* that is equal to zero. By the complementary slackness conditions we know that there exists at least one such variable. It thus remains to show the uniqueness, which we do by counting the variables that are zero in the vector (λ^*, v^*, s^*) . There are at least $m - |M|$ variables λ_i^* , $|F|$ variables s_{ij}^* corresponding to $j \in B$, and $n - |B|$ variables s_{ij}^* corresponding to $j \notin B$ that are equal to zero. Thus, in total, there are at least $(m - |M|) + |F| + (n - |B|) = m + n$ zeroes in the dual solution, where the equality follows from Lemma 3.5.1. So, these are exactly all the variables at level zero in the vector (λ^*, v^*, s^*) . Then, for each $j \notin B$ there exists exactly one variable $s_{ij}^* = 0$, and Claim (i) follows. \square

Chapter 4

Generating experimental data for the GAP

4.1 Introduction

The behaviour of a solution procedure for an optimization problem is frequently illustrated by testing it on a collection of problem instances. Conclusions about relevant characteristics of the solution procedure are drawn from such a study. A more thorough analysis may be performed by comparing the behaviour of the solution procedure, with respect to these characteristics, to other procedures on the same collection of problem instances. Typical examples of such characteristics are the computation time required by the solution procedure and the quality of the solution obtained (in case finding an optimal solution is not guaranteed). The validity of the derived conclusions strongly depends on the set of problem instances chosen for this purpose. Therefore, the collection of test problems should possess certain properties to ensure the credibility of the conclusions.

Hall and Posner [68] study test problem generations for several classes of scheduling problems. They suggest a list of desirable properties of a collection of test problems which applies to most types of deterministic mathematical programming problems. This list includes properties like robustness, consistency, and convenience. The first one requires a vast and diverse generation of test problems modeling real-life situations. This property is of special relevance when comparing the behaviour of several solution procedures. The consistency refers to invariance of the characteristics of the data generation scheme with respect to the size and the scale of the input data. The third property ensures that problem instances are easy to generate. An additional issue included by the authors under this property is the feasibility of the problem instances. Amini and Racer [1] point out that test problem instances with few feasible solutions are able to test the performance of a solution procedure better than problem instances that admits many solutions. In this chapter, we focus our attention on this issue.

Generally, the availability of real data for problem instances is very limited. A

reduced number of problem instances can bias the conclusions drawn about the behaviour of the solution procedure. Therefore, it is often desirable to generate artificial problem instances. In the literature, randomly generated problem instances are frequently used. However, the properties of the problem instances obtained by those random generators are rarely studied. We devote this chapter to the analysis of the tightness of the problem instances obtained by the random generators for the GAP proposed in the literature.

Stochastic models for the GAP have been proposed by Dyer and Frieze [38], and Romeijn and Piersma [110]. In the former paper a limited probabilistic analysis is performed, with the purpose of constructing an algorithm that solves the GAP with high probability in polynomial time (in the number of tasks). In the latter paper a tight condition on the stochastic model under which the GAP is feasible with probability one when the number of tasks n goes to infinity is derived. Furthermore, a probabilistic analysis of the optimal solution for the GAP under this model is performed, studying the asymptotic behaviour of the optimal solution value.

The stochastic model considered by Romeijn and Piersma [110] chooses the relative capacities fixed with respect to, and independent of, the requirements. In contrast, many of the random generators in the literature for the GAP define the capacities as a function of the requirements. Therefore, we propose a new stochastic model for the GAP which has as a special case the one proposed by Romeijn and Piersma [110]. A tight condition on this stochastic model under which the GAP is feasible with probability one when n goes to infinity is derived. This new stochastic model enables us to analyze the tightness of most of the random generators given for the GAP in the literature. We demonstrate that those random generators tend to create easier problem instances when the number of agents m increases, and propose a modification to the generators to overcome this defect.

The outline of this chapter is as follows. In Section 4.2 we describe a new stochastic model for the GAP, which we will call RR. We derive a tight condition on this stochastic model under which feasible problem instances are generated with probability one when the number of tasks grows to infinity, and make this condition more explicit under additional assumptions. In Section 4.3 related stochastic models are described which basically cover all the random generators proposed in the literature for the GAP. Those are analyzed by using the feasibility condition obtained for RR. In Section 4.4 some numerical results illustrate how the conclusions drawn about the performance of a greedy heuristic for the GAP differ depending on the random generator used. The results of this chapter can be found in Romeijn and Romero Morales [111].

4.2 Stochastic model for the GAP

Let the random vectors $\mathbf{A}_j = (\mathbf{A}_{1j}, \dots, \mathbf{A}_{mj})$ be i.i.d. absolutely continuous random vectors in the bounded set $[\underline{A}, \overline{A}]^m$ where \underline{A} and $\overline{A} \in \mathbb{R}_+$. Note that in the absence of remarkable differences between the agents, the fraction n/m is a rough estimate of the number of tasks assigned to each agent. The average capacity available per task can now be defined as the convex combination of two terms, each of which

represents a target size for each agent. The first one, μ_i , is fixed and independent of the requirements. The second one is equal to $\sum_{j=1}^n \mathbf{A}_{ij}/n$ which depends on the requirements at each agent. To control the tightness of the problem instances, we multiply this convex combination by a strictly positive factor δ . Thus, the capacity of agent i , say \mathbf{b}_i , is equal to

$$\mathbf{b}_i = \delta \left(\alpha_1 \mu_i + \alpha_2 \sum_{j=1}^n \mathbf{A}_{ij}/n \right) n/m$$

or

$$\mathbf{b}_i = \delta \left(\alpha_1 \mu_i n/m + \alpha_2 \sum_{j=1}^n \mathbf{A}_{ij}/m \right)$$

where δ is a strictly positive number, α_1 and α_2 are the coefficients of the convex combination, i.e., α_1 and α_2 are nonnegative and $\alpha_1 + \alpha_2 = 1$, and μ_i is a strictly positive number for each $i = 1, \dots, m$. Hereafter, we denote this stochastic model by RR. Whenever the parameters need to be known we will use the notation $\text{RR}(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A})$, where $\mu = (\mu_i)_{i=1, \dots, m}$.

If $\alpha_2 = 0$, the average capacity per task for agent i is fixed with respect to the requirements and equal to μ_i . Thus, the stochastic model addressed by Romeijn and Piersma [110] can be obtained by choosing $\alpha_2 = 0$, i.e., $\text{RR}(1, 0, \mu, \delta, \mathbf{A})$. Recall that the stochastic model considered by Romeijn and Piersma [110] is just the one proposed for the CCAP (see Section 2.2) for the particular case of the GAP. Observe that we are using the notation \mathbf{A}_{ij} instead of \mathbf{A}_j^i .

Using the concept of asymptotically equivalent stochastic models, as introduced in Section 2.2.5, we derive in Section 4.2.1 a feasibility condition for RR which, as a special case, yields the feasibility condition derived by Romeijn and Piersma [110]. In Section 4.2.2, we analyze the particular case where the requirements are independently and identically distributed according to an increasing failure rate distribution. The uniform distribution is the most widely used distribution in the random generators proposed in the literature. This is an example of an increasing failure rate distribution, and we analyze it in Section 4.2.3. Again the concept of asymptotically equivalent stochastic models will enable us to analyze the tightness of other stochastic models through the condition derived for RR.

4.2.1 Feasibility condition

Define

$$\Delta = \min_{\lambda \in S} \left(\delta/m \lambda^\top (\alpha_1 \mu + \alpha_2 \mathcal{E}(\mathbf{A}_1)) - \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i \mathbf{A}_{i1} \right) \right) \quad (4.1)$$

where S is the unit simplex in \mathbb{R}^m . Whenever the parameters need to be known we will use the notation $\Delta(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A})$.

Theorem 4.2.1 *Under RR, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$.*

Proof: First, we will show that $\text{RR}(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A})$ is asymptotically equivalent to $\text{RR}(1, 0, \alpha_1 \mu + \alpha_2 \mathcal{E}(\mathbf{A}_{\cdot 1}), \delta, \mathbf{A})$. By construction, the requirements are equally distributed in both of the models. Therefore, the first condition for asymptotic equivalence of stochastic models is satisfied (see Definition 2.2.7). Let \mathbf{b} be the vector of capacities in $\text{RR}(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A})$ and \mathbf{b}' be the vector of capacities for $\text{RR}(1, 0, \alpha_1 \mu + \alpha_2 \mathcal{E}(\mathbf{A}_{\cdot 1}), \delta, \mathbf{A})$. By the Law of the Large Numbers, we have that

$$\frac{\mathbf{b}_i}{n} \rightarrow \delta(\alpha_1 \mu_i + \alpha_2 \mathcal{E}(\mathbf{A}_{i1})) / m$$

with probability one as n goes to infinity. Moreover,

$$\frac{\mathbf{b}'_i}{n} = \delta(\alpha_1 \mu_i + \alpha_2 \mathcal{E}(\mathbf{A}_{i1})) / m.$$

Therefore the limits of the relative capacities generated by both of the models are equal, and thus the second condition for equivalence also holds.

Note that the relative capacities generated by $\text{RR}(1, 0, \alpha_1 \mu + \alpha_2 \mathcal{E}(\mathbf{A}_{\cdot 1}), \delta, \mathbf{A})$ are deterministic, and thus Theorem 2.2.4 can be applied for the particular case of the GAP. Hence, under $\text{RR}(1, 0, \alpha_1 \mu + \alpha_2 \mathcal{E}(\mathbf{A}_{\cdot 1}), \delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$. The result now follows from Proposition 2.2.8. \square

As pointed out in Chapter 2, this is an implicit condition which implies finding the optimal solution of a nonlinear minimization problem. In the next section, under additional assumptions on the requirements, we will find a more explicit expression for Δ .

4.2.2 Identical increasing failure rate requirements

In this section, we assume that the requirements are independently and identically distributed according to some increasing failure rate (IFR) distribution with support $[\underline{A}, \bar{A}]$. Recall that an IFR distribution is an absolutely continuous distribution with distribution function F and density function f , such that the failure rate (or hazard) function $f(a)/(1 - F(a))$ is an increasing function of a , see e.g. Ross [120]. Due to the fact that the requirements are identically distributed for all the agents, it is reasonable to choose $\mu_i = \mu_1$ for all $i = 1, \dots, m$. In Theorem 4.2.4, we obtain a more explicit expression for Δ under these assumptions.

A straightforward corollary of Theorem 4.1 in Piersma and Romeijn [104] will be used in the proof of Theorem 4.2.4.

Theorem 4.2.2 (cf. Piersma and Romeijn [104]) *Let $\mathbf{W}_1, \dots, \mathbf{W}_m$ be random variables, independently and identically distributed according to some IFR distribution H (with density h) on $[0, 1]$, that is, the failure (or hazard) function*

$$\frac{h(w)}{1 - H(w)}$$

is an increasing function of w . Furthermore, let $\lambda_1, \dots, \lambda_m$ be nonnegative constants satisfying $\sum_{i=1}^m \lambda_i = 1$. Define the random variables

$$\mathbf{X}_\lambda^{(m)} = \min_{i=1, \dots, m} \lambda_i \mathbf{W}_i$$

and

$$\mathbf{Y}^{(m)} = \frac{1}{m} \min_{i=1, \dots, m} \mathbf{W}_i.$$

Then

$$\mathbf{Y}^{(m)} \geq_{\text{st}} \mathbf{X}_\lambda^{(m)}$$

for all $\lambda_1, \dots, \lambda_m$ as above and $m = 1, 2, \dots$

Corollary 4.2.3 Let $\mathbf{W}_1, \dots, \mathbf{W}_m$ be i.i.d. according to some IFR distribution on $[\underline{W}, \overline{W}]$. Furthermore, let $\lambda_1, \dots, \lambda_m$ be nonnegative constants satisfying $\sum_{i=1}^m \lambda_i = 1$. Define the random variables

$$\mathbf{X}_\lambda^{(m)} = \min_{i=1, \dots, m} \lambda_i \mathbf{W}_i$$

and

$$\mathbf{Y}^{(m)} = \frac{1}{m} \min_{i=1, \dots, m} \mathbf{W}_i.$$

Then

$$\mathbf{Y}^{(m)} \geq_{\text{st}} \mathbf{X}_\lambda^{(m)}$$

for all $\lambda_1, \dots, \lambda_m$ as above and $m = 1, 2, \dots$

Proof: The proof of this result follows by scaling vector \mathbf{W} to $[0, 1]$ and applying Theorem 4.2.2.

Let H be the distribution function of \mathbf{W}_1 and h its density function. Let $\widetilde{\mathbf{W}}_i = \frac{\mathbf{W}_i - \underline{W}}{\overline{W} - \underline{W}}$. The failure rate function of random variable $\widetilde{\mathbf{W}}_i$ is equal to

$$(\overline{W} - \underline{W}) \cdot \frac{h(\underline{W} + (\overline{W} - \underline{W}) w)}{1 - H(\underline{W} + (\overline{W} - \underline{W}) w)},$$

which is an increasing function of w , since we have the composition of a linear function with positive slope with an increasing function. Therefore, the result follows for vector $\widetilde{\mathbf{W}}$, and then, for vector \mathbf{W} . \square

Theorem 4.2.4 Let \mathbf{A}_{i1} , $i = 1, \dots, m$, be i.i.d. according to an IFR distribution with support $[\underline{A}, \overline{A}]$. Assume that $\mu_i = \mu_1$ for all $i = 1, \dots, m$. Then, Δ is equal to

$$\Delta = \delta/m (\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})) - 1/m \mathcal{E} \left(\min_{i=1, \dots, m} \mathbf{A}_{i1} \right).$$

Proof: Let e be the vector in \mathbb{R}^m with all the components equal to one. By using the definition of Δ , we have that

$$\begin{aligned}\Delta &= \min_{\lambda \in S} \left(\delta/m \lambda^\top (\alpha_1 \mu + \alpha_2 \mathcal{E}(\mathbf{A}_{\cdot 1})) - \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i \mathbf{A}_{i1} \right) \right) \\ &= \min_{\lambda \in S} \left(\delta/m \lambda^\top (\alpha_1 \mu_1 e + \alpha_2 \mathcal{E}(\mathbf{A}_{11})) - \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i \mathbf{A}_{i1} \right) \right) \\ &= \min_{\lambda \in S} \left(\delta/m (\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})) - \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i \mathbf{A}_{i1} \right) \right) \quad (4.2)\end{aligned}$$

$$\begin{aligned}&= \delta/m (\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})) - \max_{\lambda \in S} \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i \mathbf{A}_{i1} \right) \\ &= \delta/m (\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})) - \mathcal{E} \left(1/m \min_{i=1, \dots, m} \mathbf{A}_{i1} \right) \quad (4.3) \\ &= \delta/m (\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})) - 1/m \mathcal{E} \left(\min_{i=1, \dots, m} \mathbf{A}_{i1} \right),\end{aligned}$$

where (4.2) follows from $\sum_{i=1}^m \lambda_i = 1$, and (4.3) from Corollary 4.2.3. \square

In this case, Theorem 4.2.1 reads as follows.

Theorem 4.2.5 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to an IFR distribution with support $[\underline{A}, \bar{A}]$ and $\mu_i = \mu_1$ for all $i = 1, \dots, m$. Under RR, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})},$$

and infeasible with probability one if this inequality is reversed.

Proof: This result follows from Theorem 4.2.1 and Theorem 4.2.4. \square

We may observe that the lower bound

$$\underline{\delta}(m) \equiv \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})}$$

on the parameter controlling the tightness of the problem instances, δ , decreases when m increases. Moreover,

$$\lim_{m \rightarrow +\infty} \underline{\delta}(m) = \frac{\underline{A}}{\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})}.$$

Through this particular case of RR, we realize that δ should depend on the number of agents m , say $\delta(m)$, in such a way that $\delta(m)$ decreases when the number of agents increases. In Section 4.3, we will deduce that the random generators proposed in the literature for the GAP are not adequate since they do not reflect this dependence on the number of agents m , but choose δ constant. Therefore, when the number of agents grows the problem instances generated are less tight.

4.2.3 Uniformly distributed requirements

Most of the random generators for the GAP proposed in the literature assume that the requirements are independently and identically distributed according to a uniform distribution. Since the uniform distribution has IFR, Theorem 4.2.5 can be applied to this particular distribution to obtain a lower bound on the parameter measuring the tightness of the problem instances, δ , with probability one when n goes to infinity. In the following result, we assume that $\mu_i = \mathcal{E}(\mathbf{A}_{11})$ for all $i = 1, \dots, m$ to impose the same target size as the random generators from the literature.

Recall that e represents the vector in \mathbb{R}^m with all the components equal to one.

Corollary 4.2.6 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to a uniform distribution with support $[\underline{A}, \bar{A}]$. Under $RR(\alpha_1, \alpha_2, \mathcal{E}(\mathbf{A}_{11})e, \delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > 2 \frac{m \cdot \underline{A} + \bar{A}}{(m+1)(\underline{A} + \bar{A})},$$

and infeasible if this inequality is reversed.

Proof: The result follows from Theorem 4.2.5 by substituting

$$\mathcal{E}\left(\min_{i=1, \dots, m} \mathbf{A}_{i1}\right) = \frac{m \cdot \underline{A} + \bar{A}}{m+1},$$

$$\mu_i = \mathcal{E}(\mathbf{A}_{11}) \text{ for all } i = 1, \dots, m, \text{ and } \mathcal{E}(\mathbf{A}_{11}) = \frac{\underline{A} + \bar{A}}{2}. \quad \square$$

As shown in Section 4.2.2, the obtained lower bound

$$\delta_U(m) = 2 \frac{m \cdot \underline{A} + \bar{A}}{(m+1)(\underline{A} + \bar{A})}$$

on the parameter controlling the tightness of the problem instances, δ , decreases when m increases and converges to $2 \cdot \underline{A}/(\underline{A} + \bar{A})$ as m goes to infinity. Moreover, $\delta_U(m)$ can be rewritten as

$$\delta_U(m) = \frac{2}{m+1} \left[1 + \frac{m-1}{1 + \frac{\bar{A}}{\underline{A}}} \right],$$

which clearly shows that it depends only on m and the ratio $\frac{\bar{A}}{\underline{A}}$. In fact, it decreases when the ratio $\frac{\bar{A}}{\underline{A}}$ increases.

In Figure 4.1, the lower bounds obtained for requirements distributed according to a uniform distribution on $[5, 25]$, $[25, 45]$, and $[1, 100]$ are plotted. Function $\delta_1(m)$ is the lower bound obtained for requirements generated uniformly on $[5, 25]$. Since $\bar{A} > \underline{A}$, we know that the ratio $\frac{\bar{A}+k}{\underline{A}+k}$, $k \geq 0$, decreases when k increases. Function $\delta_2(m)$ illustrates this fact. When interval $[5, 25]$ is shifted to $[25, 45]$ we observe that the lower bound increases since the ratio has been decreased from 5 to 9/5. Finally, function $\delta_3(m)$ is the lower bound obtained when the requirements are generated uniformly on $[1, 100]$.

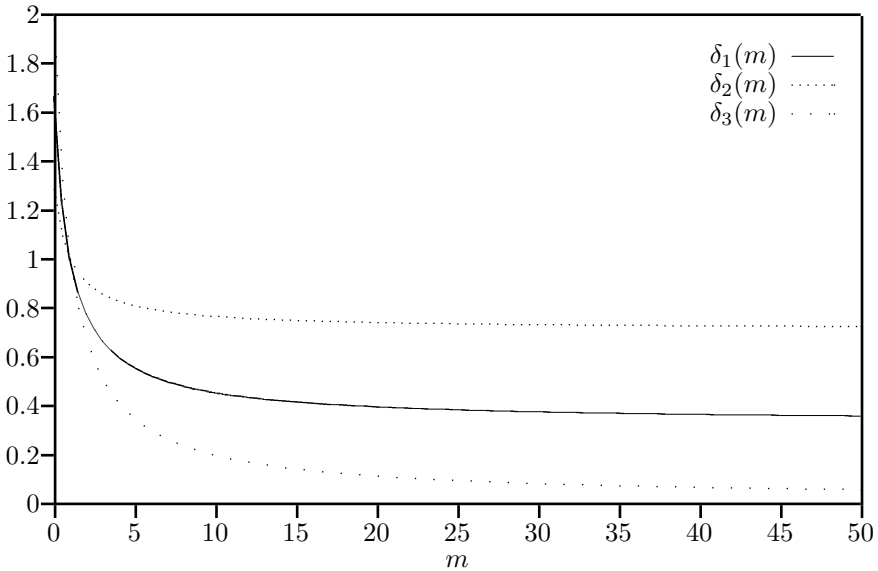


Figure 4.1: Lower bounds on the tightness

4.3 Existing generators for the GAP

4.3.1 Introduction

As is remarked by Amini and Racer [1], a problem set that admits few feasible solutions is able to test the performance of a method more so than a set that admits many solutions. So we are interested in analyzing the tightness of the problem instances of the GAP proposed in the literature.

In this section we go through most of the generators of problem instances of the GAP that can be found in the literature. Our goal is to fit each one within the stochastic model RR described in Section 4.2, or at least, to find a particular case of RR that is asymptotically equivalent to it. Through those relations we find conditions on the parameters of these stochastic models to ensure feasibility with probability one when n goes to infinity. Five new stochastic models are introduced which are generalizations of models that can be found in the literature. These stochastic models will be named by the initials of the authors who first proposed them. Throughout this section the requirements satisfy the same assumptions as in RR, i.e., $\mathbf{A}_j = (\mathbf{A}_{1j}, \dots, \mathbf{A}_{mj})$ are i.i.d. absolutely continuous random vectors in the bounded set $[\underline{A}, \overline{A}]^m$.

In Section 4.3.2, we will describe the only generator from the literature which defines the capacities as a function of both the requirements and the costs. In this case, we will assume that the vectors $\mathbf{C}_j = (\mathbf{C}_{1j}, \dots, \mathbf{C}_{mj})$ are i.i.d. absolutely continuous vectors in the bounded set $[\underline{C}, \overline{C}]^m$ where \underline{C} and $\overline{C} \in \mathbb{R}$.

We will frequently use the expression for the limit of the relative capacity gener-

ated by $\text{RR}(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A})$ obtained in Theorem 4.2.1, that is,

$$\frac{\mathbf{b}_i}{n} \rightarrow \delta (\alpha_1 \mu_i + \alpha_2 \mathcal{E}(\mathbf{A}_{i1})) / m \quad (4.4)$$

with probability one as n goes to infinity for each $i = 1, \dots, m$.

4.3.2 Ross and Soland

Let $\text{RS}(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A}, \mathbf{C})$ be the stochastic model setting the capacities to

$$\mathbf{b}_i^{\text{RS}} = \delta \left(\alpha_1 \mu_i n/m + \alpha_2 \max_{i=1, \dots, m} \sum_{j \in \mathbf{J}_i^*} \mathbf{A}_{ij} \right)$$

where δ , α_1 , α_2 , and μ satisfy the same conditions as in RR, i.e., δ is a strictly positive, α_1 and α_2 are nonnegative and $\alpha_1 + \alpha_2 = 1$, μ is a nonnegative vector, and \mathbf{J}_i^* is the random set representing the tasks for which agent i is the cheapest one, i.e.,

$$\mathbf{J}_i^* = \{j = 1, \dots, n : i = \arg \min_{s=1, \dots, m} \mathbf{C}_{sj}\} \quad (4.5)$$

where ties are broken arbitrarily.

To show the relationship between RS and RR we will use the following result.

Proposition 4.3.1 *Let $(\mathbf{A}_j, \mathbf{C}_j) \in \mathbb{R}^m \times \mathbb{R}^m$, $j = 1, \dots, n$, be i.i.d. random vectors, where \mathbf{A}_j and \mathbf{C}_j are independent. Moreover, assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d., and \mathbf{C}_{i1} , $i = 1, \dots, m$, are i.i.d.. Then,*

$$\frac{1}{n} \max_{i=1, \dots, m} \sum_{j \in \mathbf{J}_i^*} \mathbf{A}_{ij} \rightarrow \frac{1}{m} \mathcal{E}(\mathbf{A}_{i1})$$

with probability one when $n \rightarrow \infty$.

Proof: For each $i = 1, \dots, m$ and $j = 1, \dots, n$, define the auxiliary variable \mathbf{Y}_{ij} equal to \mathbf{A}_{ij} if $j \in \mathbf{J}_i^*$, and 0 otherwise. Since $(\mathbf{A}_j, \mathbf{C}_j)$ are i.i.d. and \mathbf{A}_j and \mathbf{C}_j are independent, \mathbf{Y}_{ij} , $j = 1, \dots, n$, are i.i.d.. Moreover, the expected value of \mathbf{Y}_{i1} has the following expression

$$\begin{aligned} \mathcal{E}(\mathbf{Y}_{i1}) &= \mathcal{E}(\mathbf{Y}_{i1} \mid \mathbf{C}_{i1} = \min_{s=1, \dots, m} \mathbf{C}_{s1}) \cdot \Pr(\mathbf{C}_{i1} = \min_{s=1, \dots, m} \mathbf{C}_{s1}) \\ &\quad + \mathcal{E}(\mathbf{Y}_{i1} \mid \mathbf{C}_{i1} > \min_{s=1, \dots, m} \mathbf{C}_{s1}) \cdot \Pr(\mathbf{C}_{i1} > \min_{s=1, \dots, m} \mathbf{C}_{s1}) \\ &= \mathcal{E}(\mathbf{A}_{i1} \mid \mathbf{C}_{i1} = \min_{s=1, \dots, m} \mathbf{C}_{s1}) \cdot \Pr(\mathbf{C}_{i1} = \min_{s=1, \dots, m} \mathbf{C}_{s1}) + 0 \\ &= \mathcal{E}(\mathbf{A}_{i1}) \cdot 1/m \\ &= \mathcal{E}(\mathbf{A}_{i1}) \cdot 1/m. \end{aligned}$$

Hence, for a given i , by the Strong Law of the Large Numbers we have that

$$\frac{1}{n} \sum_{j \in \mathbf{J}_i^*} \mathbf{A}_{ij} = \frac{1}{n} \sum_{j=1}^n \mathbf{Y}_{ij} \rightarrow \mathcal{E}(\mathbf{Y}_{i1}) = \mathcal{E}(\mathbf{A}_{i1})/m,$$

with probability one when $n \rightarrow \infty$. By taking the maximum over $i = 1 \dots, m$, we obtain

$$\frac{1}{n} \max_{i=1, \dots, m} \sum_{j \in \mathbf{J}_i^*} \mathbf{A}_{ij} \rightarrow \mathcal{E}(\mathbf{A}_{11})/m,$$

and then the result follows. \square

Proposition 4.3.2 *If the requirements \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d., the costs \mathbf{C}_{i1} , $i = 1, \dots, m$, are i.i.d., and \mathbf{A}_j and \mathbf{C}_j are independent, then $RS(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A}, \mathbf{C})$ is asymptotically equivalent to $RR(\alpha_1, \alpha_2, \mu, \delta, \mathbf{A})$.*

Proof: By construction, the requirements are generated in the same way in both of the models. Thus, it suffices to show that the relative capacities are equal in the limit with probability one. By Proposition 4.3.1, we have that

$$\frac{b_i^{\text{RS}}}{n} \rightarrow \delta (\alpha_1 \mu_i + \alpha_2 \mathcal{E}(\mathbf{A}_{i1}))/m$$

with probability one as $n \rightarrow \infty$ for each $i = 1, \dots, m$. The result follows from (4.4) since all $\mathcal{E}(\mathbf{A}_{i1})$ are equal to $\mathcal{E}(\mathbf{A}_{11})$. \square

The next corollary follows directly from Proposition 4.3.2 and Theorem 4.2.1.

Corollary 4.3.3 *Assume that the requirements \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d., the costs \mathbf{C}_{i1} , $i = 1, \dots, m$, are i.i.d., and vectors \mathbf{A}_j and \mathbf{C}_j are independent. Under RS, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$.*

In the particular case that the requirements are independently and identically distributed according to an IFR distribution, and $\mu_i = \mu_1$ for all $i = 1, \dots, m$, we obtain the same condition for feasibility with probability one when n goes to infinity as in Theorem 4.2.5.

Corollary 4.3.4 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to an IFR distribution with support $[\underline{A}, \bar{A}]$, \mathbf{C}_{i1} , $i = 1, \dots, m$, are i.i.d., vectors \mathbf{A}_j and \mathbf{C}_j are independent, and $\mu_i = \mu_1$ for all $i = 1, \dots, m$. Under RS, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\alpha_1 \mu_1 + \alpha_2 \mathcal{E}(\mathbf{A}_{11})},$$

and infeasible with probability one if this inequality is reversed.

Ross and Soland [118] propose the first generator for the GAP. They consider the requirements and the costs to be uniformly distributed in $[5, 25]$ and $[10, 50]$ respectively, and the capacities are set to

$$b_i = 0.6 \mathcal{E}(\mathbf{A}_{11}) n/m + 0.4 \max_{i=1, \dots, m} \sum_{j \in J_i^*} a_{ij},$$

where a_{ij} is a realization of the random variable \mathbf{A}_{ij} and J_i^* is a realization of the random set defined by (4.5). To justify this choice, they argue that one would expect random problems to be trivial when $b_i \geq \max_{i=1, \dots, m} \sum_{j \in J_i^*} a_{ij}$, and to be infeasible when $b_i < \mathcal{E}(\mathbf{A}_{11}) n/m$. (Note that in Section 4.3.4 a tighter upper bound for infeasibility with probability one when the number of tasks grows to infinity is found.) This is the particular case $RS(0.6, 0.4, \mathcal{E}(\mathbf{A}_{11}) e, 1, \mathbf{A}, \mathbf{C})$ of the model RS. Martello and Toth [88] propose the four well-known types of problem instances, A, B, C, D, which are the most used to test algorithms proposed for the GAP, see [2, 25, 32, 48, 66, 85, 101, 105, 121]. Type A is exactly the generator of Ross and Soland. However, they observe that problems of this type afford many feasible solutions. Therefore, they define a tighter kind of problem instances B, by setting b_i to 70 percent of the ones generated by type A. This is clearly still a particular case, namely $RS(0.6, 0.4, \mathcal{E}(\mathbf{A}_{11}) e, 0.7, \mathbf{A}, \mathbf{C})$, of the model RS.

Analogous to Corollary 4.2.6, we have the following corollary.

Corollary 4.3.5 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to a uniform distribution on $[\underline{A}, \overline{A}]$, \mathbf{C}_{i1} , $i = 1, \dots, m$, are i.i.d., and vectors \mathbf{A}_j and \mathbf{C}_j are independent. Under $RS(\alpha_1, \alpha_2, \mathcal{E}(\mathbf{A}_{11}) e, \delta, \mathbf{A}, \mathbf{C})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{m \cdot \underline{A} + \overline{A}}{(m+1) \mathcal{E}(\mathbf{A}_{11})},$$

and infeasible if this inequality is reversed.

In Figure 4.2 (see Section 4.3.7) the function $\underline{\delta}_U(m) = \frac{5m+25}{15(m+1)}$ corresponding to $[\underline{A}, \overline{A}] = [5, 25]$ is plotted together with horizontal lines $\delta(m) = 1$ and $\delta(m) = 0.7$.

4.3.3 Type C of Martello and Toth

Let $MTC(\delta, \mathbf{A})$ be the stochastic model setting the capacities to

$$\mathbf{b}_i^{\text{MTC}} = \delta \sum_{j=1}^n \mathbf{A}_{ij}/m$$

where δ is a strictly positive number. The stochastic model $MTC(\delta, \mathbf{A})$ is the particular case $RR(0, 1, 0, \delta, \mathbf{A})$ of RR. By Theorem 4.2.1, we know that under $MTC(\delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta(0, 1, 0, \delta, \mathbf{A}) > 0$, and infeasible with probability one if $\Delta(0, 1, 0, \delta, \mathbf{A}) < 0$. In the particular case that the requirements are independently and identically distributed according to an IFR distribution, we can obtain a more explicit condition as a special case of Theorem 4.2.5.

Corollary 4.3.6 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to an IFR distribution with support $[\underline{A}, \bar{A}]$. Under $\text{MTC}(\delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\mathcal{E}(\mathbf{A}_{11})},$$

and infeasible with probability one if this inequality is reversed.

Types C and D of the generators of Martello and Toth set b_i to

$$b_i = 0.8 \sum_{j=1}^n a_{ij} / m.$$

This is the particular case $\text{MTC}(0.8, \mathbf{A})$ of the model MTC. Type C uses the same assumptions for the requirements and the costs as types A and B described in Section 4.3.2. Type D introduces a correlation between them. In particular, the requirements are uniformly generated in $[1, 100]$ and the costs are defined as $c_{ij} = 111 - a_{ij} + u_{ij}$, where u_{ij} is uniformly generated in $(-10, 10)$. When the requirements are i.i.d. according to a uniform distribution, we obtain the same lower bound on δ as in Corollary 4.2.6. In Figure 4.2 (see Section 4.3.7) we can find the representation of the horizontal line $\delta(m) = 0.8$.

4.3.4 Trick

Let $T(\delta, \mathbf{A})$ be the stochastic model setting the capacities to

$$b_i^T = \delta \mathcal{E}(\mathbf{A}_{i1}) n / m \quad (4.6)$$

where δ is a strictly positive number. The stochastic model $T(\delta, \mathbf{A})$ is the particular case $\text{RR}(1, 0, \mu, \delta, \mathbf{A})$ of RR where $\mu_i = \mathcal{E}(\mathbf{A}_{i1})$ for each $i = 1, \dots, m$. From Theorem 4.2.1, we know that under $T(\delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta(1, 0, \mathcal{E}(\mathbf{A}_{\cdot 1}), \delta, \mathbf{A}) > 0$, and infeasible with probability one if $\Delta(1, 0, \mathcal{E}(\mathbf{A}_{\cdot 1}), \delta, \mathbf{A}) < 0$. In the particular case that the requirements are independently and identically distributed according to an IFR distribution, we can obtain a more explicit condition as a special case of Theorem 4.2.5.

Corollary 4.3.7 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to an IFR distribution on $[\underline{A}, \bar{A}]$. Under $T(\delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\mathcal{E}(\mathbf{A}_{11})},$$

and infeasible with probability one if this inequality is reversed.

Trick [131] argues that in the case of generating large problem instances, the size that makes the problem trivial, $\max_{i=1, \dots, m} \sum_{j \in J_i^*} a_{ij}$ (see Section 4.3.2), is quite large. He defines the capacities as in (4.6) where $\delta \in \{0.5, 0.75, 1\}$. He assumes the same assumptions for the requirements and the costs as Ross and Soland do. These are particular cases of T, namely $T(0.5, \mathbf{A})$, $T(0.75, \mathbf{A})$, and $T(1, \mathbf{A})$. We obtain the same lower bound on δ as in Corollary 4.2.6. In Figure 4.2 (see Section 4.3.7) we can find the representation of the horizontal lines $\delta(m) = 0.5$, $\delta(m) = 0.75$ and $\delta(m) = 1$.

4.3.5 Chalmet and Gelders

Let $CG(\alpha_1, \alpha_2, \delta, \mathbf{A})$ be the stochastic model setting the capacities to

$$b_i^{\text{CG}} = \delta \left[\alpha_1 \left(\max_{j=1, \dots, n} \mathbf{A}_{ij} - \min_{j=1, \dots, n} \mathbf{A}_{ij} \right) n/2m + \alpha_2 \min_{j=1, \dots, n} \mathbf{A}_{ij} \right]$$

where δ is a strictly positive number, α_1 and α_2 are nonnegative and $\alpha_1 + \alpha_2 = 1$. In the next proposition, the interval $[\underline{A}_i, \bar{A}_i]$ represents the support of the random variable \mathbf{A}_{i1} , for each $i = 1, \dots, m$.

Proposition 4.3.8 *Let $\mu_i = \alpha_1 \frac{\bar{A}_i - \underline{A}_i}{2}$ for each $i = 1, \dots, m$. Then, the stochastic model $CG(\alpha_1, \alpha_2, \delta, \mathbf{A})$ is asymptotically equivalent to $RR(1, 0, \mu, \delta, \mathbf{A})$.*

Proof: We have that

$$\frac{b_i^{\text{CG}}}{n} \rightarrow \delta \alpha_1 \frac{\bar{A}_i - \underline{A}_i}{2m},$$

with probability one as n goes to ∞ , for each $i = 1, \dots, m$. From (4.4), we know that the limit of the relative capacity generated by $RR(1, 0, \mu, \delta, \mathbf{A})$ for agent i is equal to $\delta \mu_i/m$, for each $i = 1, \dots, m$. The result follows now by observing that $\mu_i = \alpha_1 \frac{\bar{A}_i - \underline{A}_i}{2}$ for each $i = 1, \dots, m$. \square

We may observe that the target size $\mu_i = \alpha_1 \frac{\bar{A}_i - \underline{A}_i}{2}$ has no clear meaning in general, since it depends only on the range of the requirements. From Theorem 4.2.1, we have that under $CG(\alpha_1, \alpha_2, \delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta(1, 0, \alpha_1 \frac{\bar{A} - \underline{A}}{2}, \delta, \mathbf{A}) > 0$, and infeasible with probability one if this inequality is reversed. In the particular case that the requirements are independently and identically distributed according to an IFR distribution, we can obtain a more explicit condition as a special case of Theorem 4.2.5.

Corollary 4.3.9 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to an IFR distribution with support $[\underline{A}, \bar{A}]$. Under $CG(\alpha_1, \alpha_2, \delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\alpha_1 \frac{\bar{A} - \underline{A}}{2}},$$

and infeasible with probability one if this inequality is reversed.

Chalmet and Gelders [27] propose the following definition of the capacities

$$b_i = \delta \left[0.6 \left(\max_{j=1, \dots, n} a_{ij} - \min_{j=1, \dots, n} a_{ij} \right) n/2m + 0.4 \min_{j=1, \dots, n} a_{ij} \right].$$

They assume the same assumptions for the requirements and the costs as Ross and Soland do. This is the particular case $CG(0.6, 0.4, \delta, \mathbf{A})$ of the model CG . Since the target size imposed by this model is not reasonable, we will not analyze it further.

4.3.6 Racer and Amini

Let $RA(\delta, \mathbf{A})$ be the stochastic model setting the capacities to

$$b_i^{\text{RA}} = \max \left(\delta \sum_{j=1}^n \mathbf{A}_{ij}/m, \max_{j=1, \dots, n} \mathbf{A}_{ij} \right)$$

where δ is a strictly positive number.

Proposition 4.3.10 *The stochastic model $RA(\delta, \mathbf{A})$ is asymptotically equivalent to $RR(0, 1, 0, \delta, \mathbf{A})$.*

Proof: The result follows by observing that

$$\max \left(\delta \sum_{j=1}^n \mathbf{A}_{ij}/m, \max_{j=1, \dots, n} \mathbf{A}_{ij} \right) = \delta \sum_{j=1}^n \mathbf{A}_{ij}/m \quad \text{for each } n \geq \left\lceil \frac{m}{\delta} \cdot \frac{\bar{A}}{\underline{A}} \right\rceil.$$

□

By Theorem 4.2.1, we know that, under $RA(\delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if $\Delta(0, 1, 0, \delta, \mathbf{A}) > 0$, and infeasible with probability one if $\Delta(0, 1, 0, \delta, \mathbf{A}) < 0$. In the particular case that the requirements are independently and identically distributed according to an IFR distribution, we can obtain a more explicit condition as a special case of Theorem 4.2.5.

Corollary 4.3.11 *Assume that \mathbf{A}_{i1} , $i = 1, \dots, m$, are i.i.d. according to an IFR distribution with support $[\underline{A}, \bar{A}]$. Under $RA(\delta, \mathbf{A})$, as $n \rightarrow \infty$, the GAP is feasible with probability one if*

$$\delta > \frac{\mathcal{E}(\min_{i=1, \dots, m} \mathbf{A}_{i1})}{\mathcal{E}(\mathbf{A}_{11})},$$

and infeasible with probability one if this inequality is reversed.

Racer and Amini [105] add a type E to the list of Martello and Toth. The purpose is again correlate the requirements and the costs. The requirements are set to $a_{ij} = 1 - 10 \ln u_{ij}$, the costs to $c_{ij} = \frac{100}{a_{ij}} - 10 v_{ij}$ where \mathbf{U}_{ij} and \mathbf{V}_{ij} are uniformly distributed on $(0, 1)$, and the capacities to

$$b_i = \max \left(0.8 \sum_{j=1}^n a_{ij}/m, \max_{j=1, \dots, n} a_{ij} \right)$$

for each $i = 1, \dots, m$. This model is a particular case of RA, namely $RA(0.8, \mathbf{A})$. Since this model generates the same capacities as MTC when the number of tasks is large enough we will not analyze it further.

4.3.7 Graphical comparison

Figure 4.2 gives us an idea about the tightness of the problem instances generated by Ross and Soland, Martello and Toth, and Trick. We may recall that the lower bound obtained to generate feasible problem instances with probability one when the number of tasks grows to infinity is the same for all of them, and it is named $\underline{\delta}_U(m)$ in Figure 4.2. The other functions plotted are the horizontal lines corresponding to the constant values of δ used by each of the mentioned random generators from the literature, i.e., $rs(m) = 1$ and $mtb(m) = 0.7$ are the tightness imposed by Ross and Soland, and Martello and Toth for model RS, $mtc(m) = 0.8$ is the one imposed by Martello and Toth for the model MTC, and $t1(m) = 1$, $t2(m) = 0.75$, and $t3(m) = 0.5$ are the ones imposed by Trick for the model T.

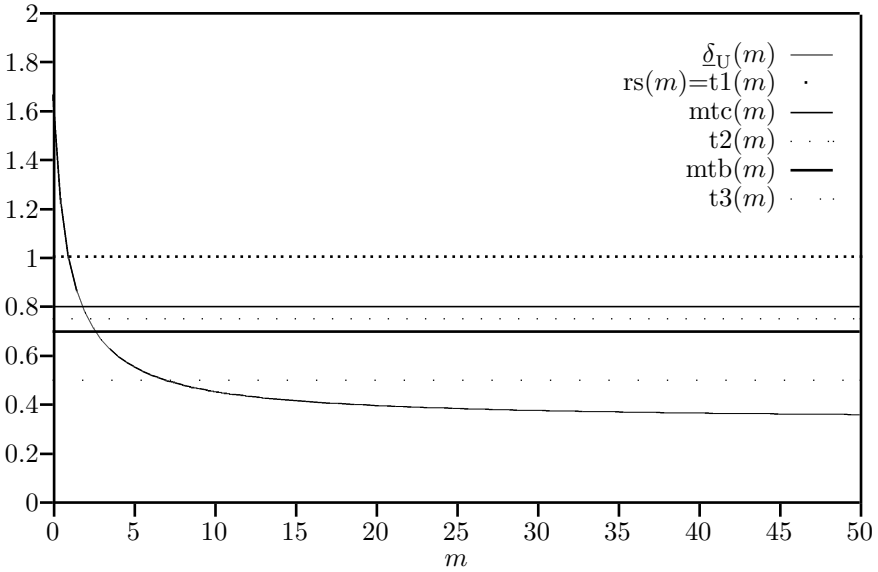


Figure 4.2: Tightness of the generators in the literature

4.4 Numerical illustrations

4.4.1 Introduction

In this section we illustrate the theoretical results from this chapter by comparing the conclusions drawn about the behaviour of a solution procedure for the GAP, using random problem instances generated by two models that are widely used in the literature: the model of Ross and Soland and model B of Martello and Toth (see Section 4.3.2), and a comparable model from the class RR.

The solution procedure was described in Section 2.3 for the CCAP and is the combination of a greedy heuristic for the CCAP and two local exchange procedures. The

heuristic that we will use in this section is a member of the class of greedy heuristics for the CCAP (see Section 2.3.1), for the particular case of the GAP. In Chapter 5, we will show asymptotic feasibility and optimality in the probabilistic sense of this heuristic under mild conditions. The two local exchange procedures for the CCAP (see Section 2.3.2) have been added to improve the quality of the solution for the GAP obtained by the greedy heuristic. This section is not intended to investigate the performance of this solution procedure, but to show how the conclusions that can be drawn about it depend on the model used for the generation of random problem instances, thereby illustrating the desirability of using a comparable set of problem instances. Note that we do not intend to recommend specific values for the tightness parameter for the models. Clearly, different contexts may call for the use of problem instances that are either very tight or very loose, and the choice that is made with respect to the tightness of problem instances may very well influence which is the most effective heuristic to be used. However, it will be clear that in any case the use of a *consistent* set of problem instances among problem sizes is important.

4.4.2 The solution procedure for the GAP

Martello and Toth [88] propose a class of greedy heuristics widely used for the GAP which has been generalized in Section 2.3.1 for the CCAP. As an input, this greedy heuristic needs a pseudo-cost function $f(i, j)$ which evaluates the assignment of task j to agent i . Martello and Toth [88] suggest several choices for the pseudo-cost function $f(i, j)$. More precisely, they claim that good computational results were obtained with $f(i, j) = c_{ij}$, $f(i, j) = a_{ij}$, $f(i, j) = a_{ij}/b_i$, and $f(i, j) = (c_{ij} - t_j)/a_{ij}$ where $t_j > \max_{i=1, \dots, m} c_{ij}$ for each $j = 1, \dots, n$. The intention of these pseudo-cost functions is very clear. The first one defines the most desirable agent as the cheapest one, and the second and the third ones consider the best agent as the one requiring the least capacity (absolutely or relatively). The last pseudo-cost function tries to combine both costs and requirements when evaluating a possible assignment. Using this idea, we will propose in Chapter 5 the class of the pseudo-cost functions

$$f(i, j) = c_{ij} + \lambda_i a_{ij},$$

where $\lambda \in \mathbb{R}_+^m$, to jointly take into account the fact that it is desirable to assign a task to an agent with minimal cost and minimal capacity usage. There we will study the behaviour of the heuristic for $\lambda_i = \lambda_i^*$ where λ_i^* is the optimal dual multiplier of the i -th capacity constraint of the LP-relaxation of the GAP. Observe that these capacity constraints have been reformulated as \geq -constraints, so that the dual subvector is nonnegative. (Clearly, if the LP-relaxation of the GAP is infeasible, so is the GAP. Therefore, the pseudo-cost function is well-defined.) Under the stochastic model proposed by Romeijn and Piersma [110], asymptotic feasibility and optimality with probability one of the greedy heuristic will be proved for this pseudo-cost function, which we will therefore use in the remainder of this chapter. We have also chosen $\lambda_i = \lambda_i^*$ for the numerical illustrations in this section.

We try to improve the current solution for the GAP given by the greedy heuristic with the two local exchange procedures for the CCAP proposed in Section 2.3.2.

Recall that the first local exchange procedure tries to assign the tasks where the greedy heuristic failed. Given a non-assigned task j , the assignment to agent i is measured by $r(i, j)$ and the best agent, say i_j , is defined as the one minimizing $r(i, j)$. Those tasks are then assigned to their most desirable agent in decreasing order of $r(i_j, j)$, either directly when agent i_j has sufficient capacity available, or through a feasible exchange, if one can be found. We have chosen $r(i, j) = a_{ij}$, i.e., the most desirable agent for task j is the one requiring the least resource. Finally, we try to improve the objective value of the current solution with the second local exchange procedure. There, the possible pairs of exchanges of tasks (ℓ, p) are considered in decreasing order of

$$(f(i_\ell, \ell) + f(i_p, p)) - (f(i_\ell, p) + f(i_p, \ell))$$

where i_ℓ and i_p are the agents to which tasks ℓ and p are assigned in the current solution and $f(i, j) = c_{ij} + \lambda_i^* a_{ij}$ is the pseudo-cost function used by the greedy heuristic.

4.4.3 Computational results

In this section we test the performance of the greedy heuristic and the local exchange procedures proposed in Section 4.4.2 on random problem instances of the GAP, using the following three models:

1. the stochastic model of Ross and Soland (Martello and Toth's type A), i.e., $\text{RS}(0.6, 0.4, \mathcal{E}(\mathbf{A}_{11}) e, 1, \mathbf{A}, \mathbf{C})$;
2. Martello and Toth's type B, i.e., $\text{RS}(0.6, 0.4, \mathcal{E}(\mathbf{A}_{11}) e, 0.7, \mathbf{A}, \mathbf{C})$; and
3. an RR generator comparable to these models, namely the stochastic model $\text{RR}\left(0.6, 0.4, \mathcal{E}(\mathbf{A}_{11}) e, 2.1 \cdot \frac{5m+25}{15(m+1)}, \mathbf{A}\right)$.

The multiplier 2.1 in RR is chosen so that problem instances generated using that model approach the problem instances generated using Martello and Toth's type B as the number of agents m grows to infinity. For completeness sake, we recall here the capacities corresponding to the three models mentioned above:

$$\begin{aligned} b_i^1 &= 0.6 \mathcal{E}(\mathbf{A}_{11}) n/m + 0.4 \max_{i=1, \dots, m} \sum_{j \in J_i^*} a_{ij} \\ b_i^2 &= 0.7 \left(0.6 \mathcal{E}(\mathbf{A}_{11}) n/m + 0.4 \max_{i=1, \dots, m} \sum_{j \in J_i^*} a_{ij} \right) \\ b_i^3 &= 2.1 \cdot \frac{5m+25}{15(m+1)} \left(0.6 \mathcal{E}(\mathbf{A}_{11}) n/m + 0.4 \frac{1}{m} \sum_{j=1}^n a_{ij} \right) \end{aligned}$$

where J_i^* is a realization of the random set defined by (4.5). As is most common in the literature, the requirements a_{ij} were generated uniformly between 5 and 25, and

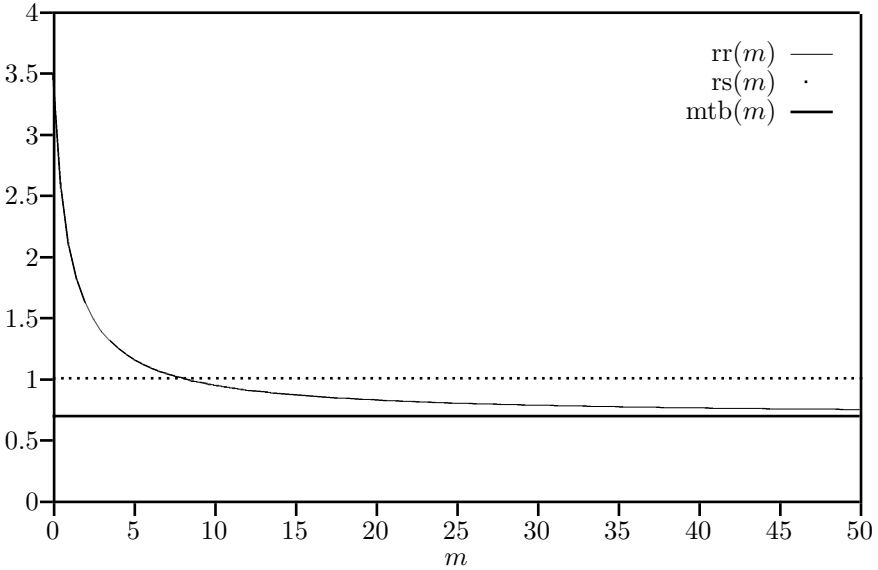


Figure 4.3: Tightness of the proposed problem instances

the costs c_{ij} uniformly between 10 and 50. In Figure 4.3, the tightness imposed by the three models are plotted together, where the notation is similar to Figure 4.2.

For the numerical experiments, the number of agents was varied from 5 to 50, and the number of tasks was chosen to be either $15m$ or $25m$. For each problem size, 100 problem instances were generated. All LP-relaxations were solved using CPLEX 6.5 [33].

Figures 4.4 and 4.5 show the behaviour of the average fraction of problem instances in which a feasible solution could be found in the first phase of the heuristic (i.e., without using the local exchange procedures to find a feasible solution or a solution with a better objective function value). We observe that this fraction increases with the number of agents for Martello and Toth's type B generator. Ross and Soland's generator seems to generate relatively easy problem instances (for which the first phase of the heuristic always finds a feasible solution), whereas the RR generator shows a modest and fairly stable number of infeasibilities.

Figures 4.6–4.9 show the behaviour of the average error bound (measured as the percentage on which the heuristic value exceeds the LP-relaxation value) as the number of agents increases. In Figures 4.6 and 4.7, the result of the greedy heuristic, including the local exchange procedure for finding a feasible solution, is shown, whereas in Figures 4.8 and 4.9 the result is shown when using, in addition, the local exchange procedure for improving the objective function value. Using Ross and Soland's and Martello and Toth's type B generators, the main conclusion would be that the heuristic works better for larger problem instances than for smaller problem instances. In addition, using Ross and Soland's generator, one could conclude

that the heuristic finds the optimal solution almost always. The theoretical results in Sections 4.2 and 4.3 show that this behaviour is not due to the characteristics of the heuristic, but due to the characteristics of the generated problem instances. In particular, as the number of agents increases, the capacity constraints are becoming less binding, making the problem instances easier. Using the generator RR, which yields problem instances that are comparable among different numbers of agents, we reach the conclusion that the heuristic performs quite well for small problem instances, with a modest increase in relative error as the size of the problem instances increases.

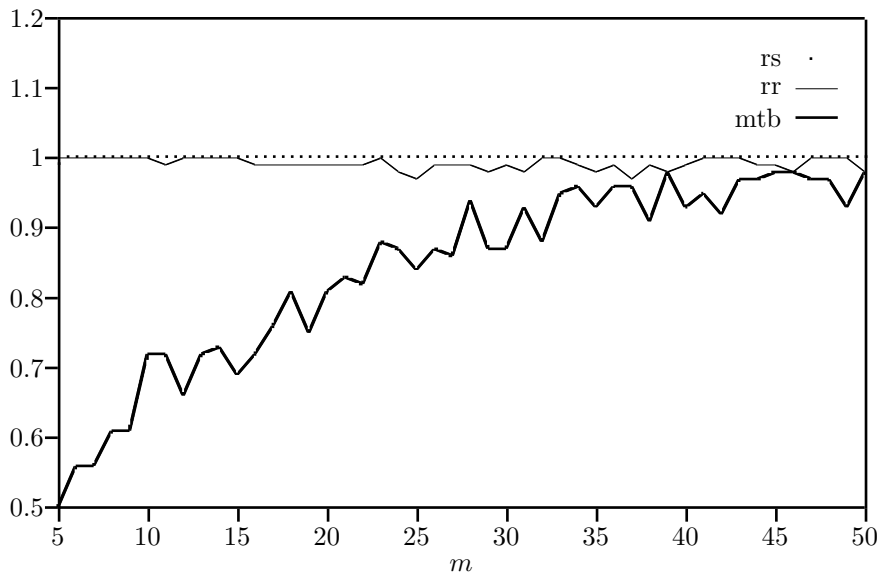


Figure 4.4: Average fraction of feasible solutions in first phase, $n = 15m$

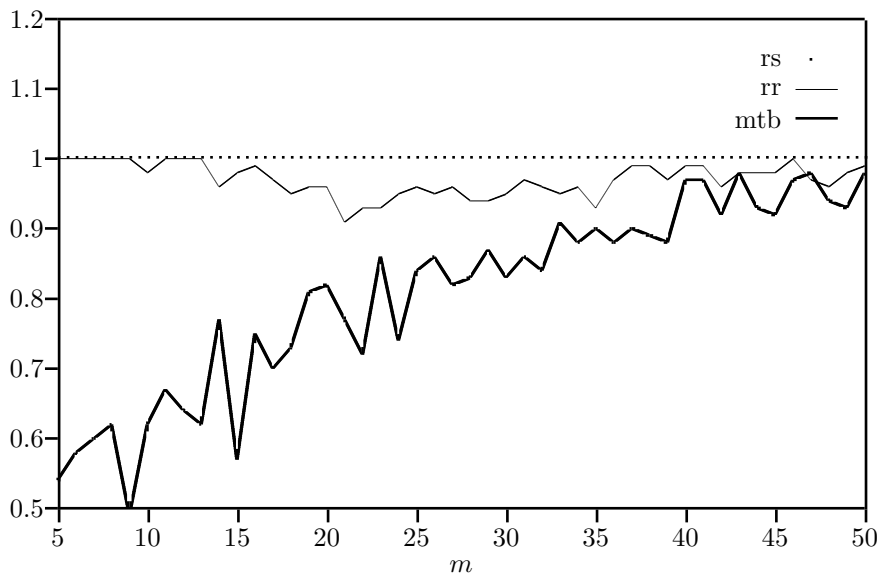


Figure 4.5: Average fraction of feasible solutions in first phase, $n = 25m$

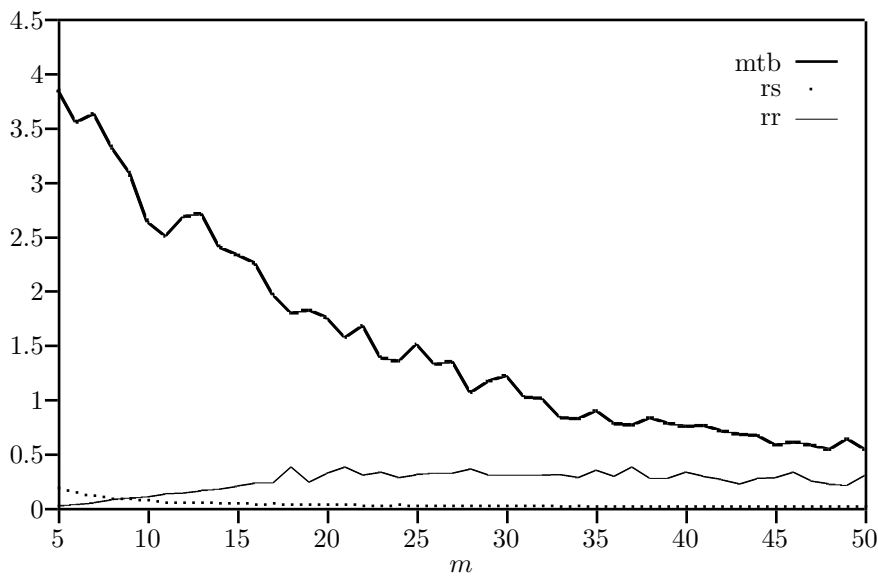


Figure 4.6: Average error (including heuristic improving feasibility), $n = 15m$

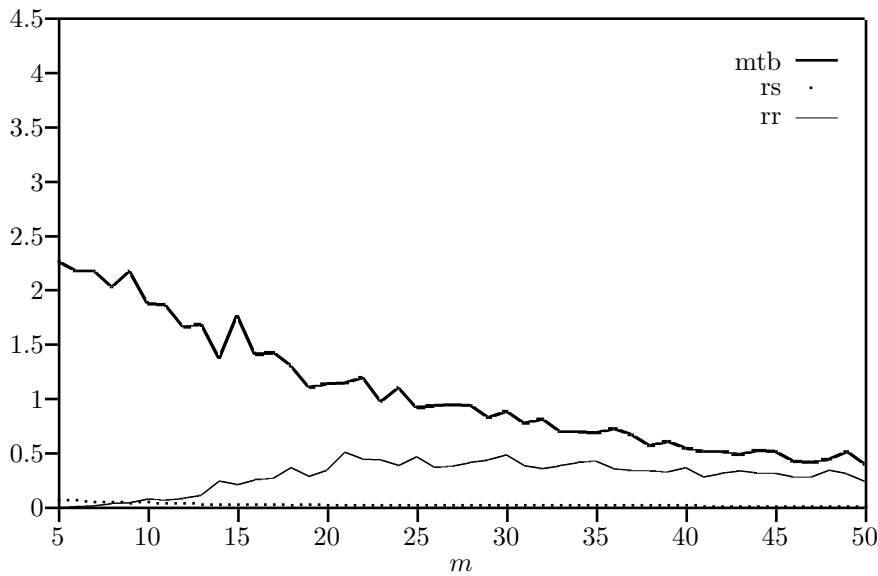


Figure 4.7: Average error (including heuristic improving feasibility), $n = 25m$

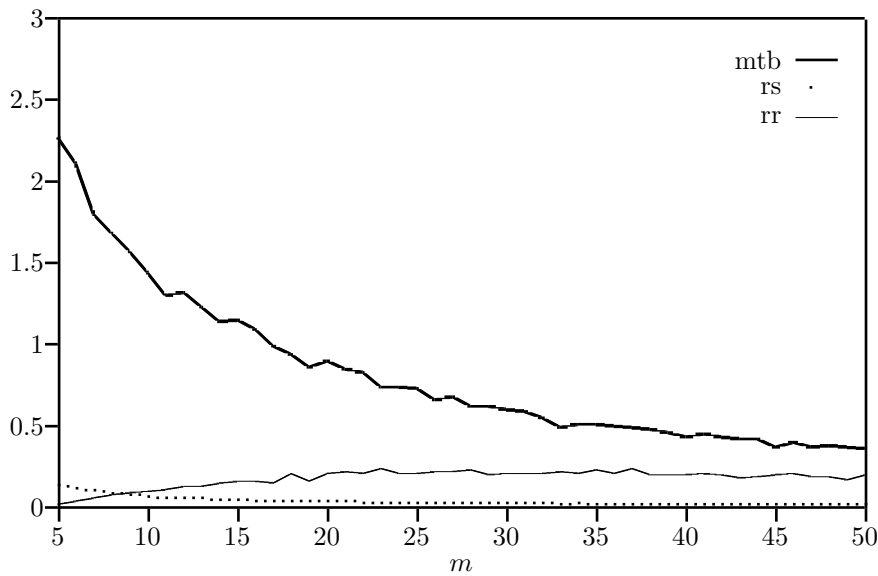


Figure 4.8: Average error (including heuristic improving objective), $n = 15m$

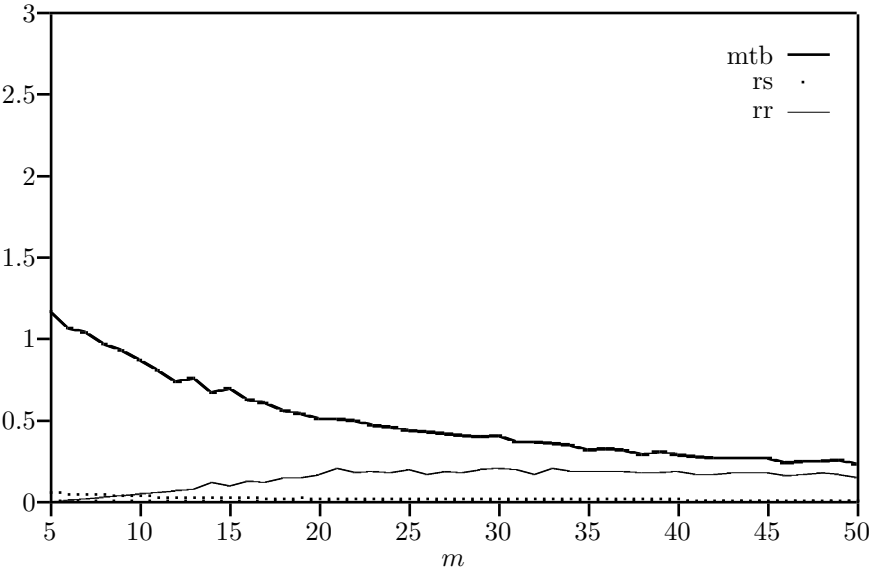


Figure 4.9: Average error (including heuristic improving objective), $n = 25m$

Chapter 5

Asymptotically optimal greedy heuristics for the GAP

5.1 A family of pseudo-cost functions

The \mathcal{NP} -Hardness of the Generalized Assignment Problem (GAP) suggests that solving large problem instances to optimality may require a substantial computational effort. Therefore, heuristic approaches are often used instead of exact procedures when the decision maker is satisfied with a *good*, rather than an *optimal*, solution. As described in Chapter 3, there is an extensive amount of literature devoted to heuristic approaches for the GAP. They are based on its LP-relaxation, Lagrangean relaxations, an equivalent set partitioning formulation, meta-heuristics, and greedy approaches. The performance of those heuristic approaches is frequently illustrated by testing them on real-life problem instances or on a collection of randomly generated problem instances. However, less attention is devoted to the study of theoretical properties of those heuristics approaches.

The most widely used class of greedy heuristics for the GAP has been proposed by Martello and Toth [88]. In Chapter 2 we have generalized this class for the class of convex capacitated assignment problems. We refresh briefly the basic idea of this greedy heuristic. The possible assignment of a task to an agent is evaluated by a pseudo-cost function $f(i, j)$ and the desirability of assigning a task is measured by the difference between the second smallest and the smallest values of $f(i, j)$. The tasks are assigned to their best agents in decreasing order of the desirability. Along the way, the remaining capacities available at the agents decrease and some of them may not be able to handle some of the tasks. Therefore, the values of the desirabilities must be updated.

Martello and Toth [88] claim that their computational results indicate that the following pseudo-cost functions are good choices:

(i) $f(i, j) = c_{ij}$,

(ii) $f(i, j) = a_{ij}$,

(iii) $f(i, j) = a_{ij}/b_i$, and

(iv) $f(i, j) = (c_{ij} - t_j)/a_{ij}$

where $t_j > \max_{i=1, \dots, m} c_{ij}$ for each $j = 1, \dots, n$. The motivation for choosing the pseudo-cost function (i) is that it is desirable to assign a task to an agent that can process it as cheaply as possible, and the motivation for the pseudo-cost functions (ii) and (iii) is that it is desirable to assign a task to an agent that can process it using the least (absolute or relative) capacity. The pseudo-cost function (iv) tries to consider the effects of the previous pseudo-cost functions jointly. (Observe that the definition of this pseudo-cost function depends on the parameters t_j which do not have a clear meaning.)

As in the pseudo-cost function (iv), we would like to take into account at the same time the fact that it is desirable to assign a task to an agent with minimal cost and minimal requirement of capacity. In order to achieve this, we define the family of pseudo-cost functions

$$\{f_\lambda(i, j) : \lambda \in \mathbb{R}_+^m\}$$

where

$$f_\lambda(i, j) = c_{ij} + \lambda_i a_{ij}.$$

Particular choices of the vector of multipliers λ yield or approximate some of the pseudo-cost functions proposed by Martello and Toth [88]. Note that if $\lambda_i = 0$ for all $i = 1, \dots, m$, we obtain the pseudo-cost function (i). Furthermore, if $\lambda_i = M$ for all $i = 1, \dots, m$, we approach the pseudo-cost function (ii) as M grows large, whereas if $\lambda_i = M/b_i$ for all $i = 1, \dots, m$ we approach the pseudo-cost function (iii) as M increases.

This family of pseudo-cost functions defines a class of greedy heuristics for the GAP. We analyze the performance of this class using a similar approach as for the Multi-Knapsack Problem (see Meanti et al. [94] and Rinnooy Kan, Stougie and Vercellis [108]). As for the probabilistic analysis of the GAP, the fact that not all instances of the problem are feasible creates significant challenges.

Recall that the LP-relaxation of the GAP, (LPR), and its dual programming problem, (D), were formulated in Section 3.5. Recall also that x^{LPR} denotes an optimal solution for (LPR), B the set of aplit assignments of x^{LPR} , and x^{G} the (partial) solution for the GAP given by the greedy heuristic.

The outline of this chapter is as follows. In Section 5.2 we give a geometric interpretation of the family of greedy heuristics. In Section 5.3 we prove that, for a fixed number of agents, the best set of multipliers can be found in polynomial time. In Section 5.4 we show that, for large problem instances (as measured by the number of tasks), the greedy heuristic finds a feasible and optimal solution with probability one with the optimal dual multipliers of the capacity constraints in the LP-relaxation of the GAP. Moreover, conditions are given under which there exists a unique vector of multipliers, only depending on the number of agents and the probabilistic model for the parameters of the problem, so that the corresponding heuristic is asymptotically feasible and optimal. Finally, Section 5.5 presents some numerical results to illustrate the behaviour of the greedy heuristic described in Section 5.4.2. Similar results can be found in Romeijn and Romero Morales [115].

5.2 Geometrical interpretation

In this section we will show how the greedy heuristic with the pseudo-cost function f_λ , $\lambda \in \mathbb{R}_+^m$, can be interpreted geometrically. To this end, define, for each task j , a set of $(m-1) \cdot m$ points $P^{jis} \in \mathbb{R}^{m+1}$ ($i, s = 1, \dots, m, s \neq i$) as follows:

$$(P^{jis})_\ell = \begin{cases} a_{ij} & \text{if } \ell = i \\ -a_{sj} & \text{if } \ell = s \\ c_{ij} - c_{sj} & \text{if } \ell = m+1 \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, define a hyperplane in \mathbb{R}^{m+1} with normal vector $(\lambda, 1)$, i.e., a hyperplane of the form

$$\left\{ p \in \mathbb{R}^{m+1} : \sum_{\ell=1}^m \lambda_\ell p_\ell + p_{m+1} = R \right\} \quad (5.1)$$

where $R \in \mathbb{R}$. Observe that this hyperplane passes through the point P^{jis} if

$$\begin{aligned} R &= \lambda_i a_{ij} - \lambda_s a_{sj} + c_{ij} - c_{sj} \\ &= f_\lambda(i, j) - f_\lambda(s, j). \end{aligned}$$

Therefore, if agent i is preferred over agent s when assigning task j with respect to the pseudo-cost function f_λ (i.e., $f_\lambda(i, j) < f_\lambda(s, j)$) then the point P^{jis} lies *below* the hyperplane of the form (5.1) with $R = 0$, whereas the point P^{jsi} lies *above* it.

Now let R be a (negative) constant such that none of the points P^{ijs} lies in the halfspace

$$\left\{ p \in \mathbb{R}^{m+1} : \sum_{\ell=1}^m \lambda_\ell p_\ell + p_{m+1} \leq R \right\} \quad (5.2)$$

and for the moment disregard the capacity constraints of the agents. When R is increased from this initial value, the corresponding halfspace starts containing points P^{jis} . The interpretation of this is that whenever a point P^{jis} is reached by the hyperplane defining the halfspace, agent i is preferred over agent s when assigning task j with respect to the pseudo-cost function f_λ . As soon as the halfspace contains, for some task j and some agent i , all points P^{jis} ($s = 1, \dots, m; s \neq i$), agent i is preferred to all other agents, and task j is assigned to agent i .

Now let us see in what order the tasks are assigned to agents. If for some task j and some agent i all points of the form P^{jis} are contained in the halfspace (5.2), then

$$R \geq \max_{s=1, \dots, m; s \neq i} (f_\lambda(i, j) - f_\lambda(s, j)).$$

The first time this occurs for some agent i is if

$$R = \min_{i=1, \dots, m} \max_{s=1, \dots, m; s \neq i} (f_\lambda(i, j) - f_\lambda(s, j))$$

or, equivalently,

$$\begin{aligned} R &= - \max_{i=1,\dots,m} \min_{s=1,\dots,m; s \neq i} (f_\lambda(s, j) - f_\lambda(i, j)) \\ &= -\rho_j. \end{aligned}$$

Finally, the first task for which this occurs is the task for which the above value of R is minimal, or equivalently, for which ρ_j is maximal. Thus, when capacity constraints are not considered, the movement of the hyperplane orders the tasks in the same way as the desirabilities ρ_j .

The modification of the geometric version of the greedy heuristic to include capacity constraints is straightforward. After making an assignment the remaining capacities at the agents decrease. Therefore, some of the future assignments may not be possible anymore due to shortage of capacity. In this case the greedy heuristic would recalculate the desirabilities. In the following we will show that this can be seen as a movement backwards of the hyperplane. Recall that for a given set of multipliers λ , the position of the hyperplane is determined by R . Let R_0 be the current position of the hyperplane. Suppose then that task j has not been assigned yet. Suppose also that after the last assignment the capacity of agent i has been decreased so that task j cannot be assigned anymore to agent i . In the geometric algorithm, this assignment is associated with points P^{jis} and P^{jsi} ($s = 1, \dots, m$ and $s \neq i$). Therefore, we need to eliminate all those points since they are meaningless. The same follows for all the tasks which have not been assigned yet. Since the number of points in \mathbb{R}^{m+1} associated with the assignments of those tasks has changed, the hyperplane needs to go back until no remaining point is below the hyperplane. Hence, the recalculation of the desirabilities due to adjustments in the remaining capacity can be seen as a step backwards of the hyperplane. Note that if at some point of time all points corresponding to a task have been removed, this task cannot be assigned feasibly by the greedy heuristic.

A step backwards of the hyperplane, or equivalently, a recalculation of the desirabilities can be very expensive when the number of tasks is large. Fortunately, the number of times that the hyperplane must make a step backwards only depends on the set of feasible agents for each task j which remains to be assigned. The feasibility of an agent is only an issue when its remaining capacity is below $\max_{i=1,\dots,m; j=1,\dots,n} a_{ij}$, and then the hyperplane only needs to make a step backwards when, after making an assignment, the remaining capacity of the corresponding agent is below $\max_{i=1,\dots,m; j=1,\dots,n} a_{ij}$. This happens at most

$$\left\lceil \frac{\max_{i=1,\dots,m; j=1,\dots,n} a_{ij}}{\min_{i=1,\dots,m; j=1,\dots,n} \{a_{ij} : a_{ij} > 0\}} \right\rceil$$

for each agent. (Observe that assigning a task with requirement 0 does not alter the current desirabilities.) Thus the number of times that the hyperplane makes a step backwards is no more than

$$m \left\lceil \frac{\max_{i=1,\dots,m; j=1,\dots,n} a_{ij}}{\min_{i=1,\dots,m; j=1,\dots,n} \{a_{ij} : a_{ij} > 0\}} \right\rceil.$$

5.3 Computational complexity of finding the best multiplier

The performance of the greedy heuristic depends on the choice of a nonnegative vector $\lambda \in \mathbb{R}^m$. Obviously, we would like to choose this vector λ in such a way that the solution obtained is the one with the smallest objective function value attainable by the class of greedy heuristics. We make the dependence on the solution found by the greedy heuristic on λ explicit by denoting this solution by $x_{ij}^G(\lambda)$. Then define for each vector $\lambda \in \mathbb{R}_+^m$

$$z^G(\lambda) = \begin{cases} \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}^G(\lambda) & \text{if the greedy heuristic is feasible for } \lambda \\ \infty & \text{otherwise.} \end{cases}$$

If there exists a vector $\lambda \in \mathbb{R}_+^m$ with $z^G(\lambda) < \infty$ (in other words, the greedy heuristic gives a feasible solution for the GAP for λ), we can define the best vector, $\tilde{\lambda}$, as the minimizer of $z^G(\lambda)$ over all the nonnegative vectors $\lambda \in \mathbb{R}^m$ (if this minimum exists), i.e.,

$$z^G(\tilde{\lambda}) = \min_{\lambda \in \mathbb{R}_+^m} z^G(\lambda).$$

The following result shows how we can find the best set of multipliers, or decide that no choice of multipliers yields a feasible solution, in polynomial time if the number of agents m is fixed. (See Rinnooy Kan, Stougie and Vercellis [108] for an analogous result for a class of generalized greedy heuristics for the Multi-Knapsack Problem.)

Theorem 5.3.1 *If the number of agents m in the GAP is fixed, there exists a polynomial time algorithm to determine an optimal set of multipliers, or to decide that no vector $\lambda \in \mathbb{R}_+^m$ exists such that the greedy heuristic finds a feasible solution for the GAP.*

Proof: Each vector $\lambda \in \mathbb{R}_+^m$ induces an ordering of the points P^{jis} , and thus an assignment of tasks to agents as well as an ordering of these assignments. Each of these orderings is given by a hyperplane in \mathbb{R}^{m+1} , and thus we need to count the number of hyperplanes giving different orderings. Those can be found by shifting hyperplanes in \mathbb{R}^{m+1} . The number of possible orderings is $O(n^{m+1} \log n)$ (see Rinnooy Kan, Stougie and Vercellis [108] and Lenstra et al. [84]). For each order obtained, the greedy heuristic requires $O(n^2)$ time to compute the solution for the GAP or to decide that no feasible solution could be found. Then, all the possible solutions can be found in $O(n^{m+3} \log n)$ time. In the best case, when at least there exists a vector $\lambda \in \mathbb{R}_+^m$ giving a feasible solution, we need $O(\log(n^{m+3} \log n)) = O(\log n)$ time to select the best set of multipliers. Thus, in $O(n^{m+3} \log n)$ we can find the best set of multipliers, or decide that the greedy heuristic is infeasible for each $\lambda \in \mathbb{R}_+^m$. \square

5.4 Probabilistic analysis

5.4.1 A probabilistic model

In this section we will probabilistically analyze the asymptotic behaviour of two members of the class of the greedy heuristics given in the introduction. We will generate problem instances of the GAP with the stochastic model proposed by Romeijn and Piersma [110]. Let the random vectors $(\mathbf{A}_j, \mathbf{C}_j)$ ($j = 1, \dots, n$) be i.i.d. according to an absolutely continuous probability distribution in the bounded set $[\underline{A}, \bar{A}]^m \times [\underline{C}, \bar{C}]^m$ where $\underline{A}, \bar{A}, \underline{C}$ and $\bar{C} \in \mathbb{R}_+$. Furthermore, let b_i depend linearly on n , i.e., $b_i = \beta_i n$, for positive constants $\beta_i \in \mathbb{R}_+$. Throughout this section we will assume that $\underline{A} > 0$ and $\beta_i < \mathcal{E}(\mathbf{A}_{i1})$ for each $i = 1, \dots, m$.

Recall that we have proposed in Chapter 2 a stochastic model for the parameters defining the feasible region of the CCAP in a similar fashion as the stochastic model of Romeijn and Piersma [110] for the GAP. As shown by Romeijn and Piersma [110], feasibility of the problem instances of the GAP is not guaranteed under the above stochastic model, even for (LPR). From Theorem 2.2.4, we know that the following assumption ensures feasibility of the GAP with probability one as n goes to infinity.

Assumption 5.4.1 *The excess capacity*

$$\Delta = \min_{\lambda \in S} \left(\lambda^\top \beta - \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i \mathbf{A}_{i1} \right) \right)$$

(where S is the unit simplex) is strictly positive.

Recall that this result was shown by Romeijn and Piersma [110] for the particular case of the GAP, and generalized in Chapter 2 for the CCAP.

Under feasibility of the GAP, some results on the convergence of the normalized optimal solution value of (LPR) and the GAP are derived by Romeijn and Piersma [110]. Let \mathbf{Z}_n be the random variable representing the optimal solution value of the GAP, and $\mathbf{Z}_n^{\text{LPR}}$ be the optimal solution value of (LPR). Let $\mathbf{X}_n^{\text{LPR}}$ be the random vector representing the optimal solution vector for (LPR).

Theorem 5.4.2 (cf. Romeijn and Piersma [110]) *The normalized optimal solution value of (LPR), $\frac{1}{n} \mathbf{Z}_n^{\text{LPR}}$, tends to*

$$\theta \equiv \max_{\lambda \geq 0} \left(\mathcal{E} \left(\min_{i=1, \dots, m} (\mathbf{C}_{i1} + \lambda_i \mathbf{A}_{i1}) \right) - \lambda^\top \beta \right)$$

with probability one when n goes to infinity.

Under an additional assumption, Romeijn and Piersma [110] show that the normalized optimal value of the GAP converges to the same constant θ . Recall that \mathbf{e} denotes the vector in \mathbb{R}^m whose components are all equal to one.

Theorem 5.4.3 (cf. Romeijn and Piersma [110]) *Define the function $\psi : \mathbb{R} \rightarrow \mathbb{R}$ as*

$$\psi(x) = \min_{\lambda \geq x\mathbf{e}} \left(\lambda^\top \beta - \mathcal{E} \left(\min_{i=1, \dots, m} (\mathbf{C}_{i1} + \lambda_i \mathbf{A}_{i1}) \right) \right).$$

Then, $\psi'_+(0)$, the right derivative of ψ at 0, exists. Moreover, if $\psi'_+(0) > 0$, then

$$\mathbf{Z}_n \leq \mathbf{Z}_n^{\text{LPR}} + (\overline{C} - \underline{C}) \cdot m$$

with probability one as $n \rightarrow \infty$.

We will then assume the following.

Assumption 5.4.4 *The right derivative of $\psi : \mathbb{R} \rightarrow \mathbb{R}$ at 0 is strictly positive, where*

$$\psi(x) = \min_{\lambda \geq x\mathbf{e}} \left(\lambda^\top \beta - \mathcal{E} \left(\min_{i=1, \dots, m} (\mathbf{C}_{i1} + \lambda_i \mathbf{A}_{i1}) \right) \right).$$

The proof of Theorem 5.4.3 is based on showing that, under Assumption 5.4.4, the normalized sum of the slacks of the capacity constraints of the optimal solution for (LPR) is eventually strictly positive. Since we will make explicit use of this result, we state it as a theorem.

Theorem 5.4.5 *(cf. Romeijn and Piersma [110]) Under Assumption 5.4.4, it holds*

$$\sum_{i=1}^m \beta_i - \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n \mathbf{A}_{ij} \mathbf{X}_{ij}^{\text{LPR}} > 0$$

with probability one as $n \rightarrow \infty$.

When the requirements are agent-independent, i.e., for each $j = 1, \dots, n$, $\mathbf{A}_{ij} = \mathbf{D}_j$ for all $i = 1, \dots, m$, Romeijn and Piersma [110] have shown that Assumptions 5.4.1 and 5.4.4 coincide. In particular, they have proved that these assumptions are equivalent to the condition $\mathcal{E}(\mathbf{D}_1) < \sum_{i=1}^m \beta_i$.

Theorem 5.4.6 *(cf. Romeijn and Piersma [110]) If the requirements are agent-independent, Assumptions 5.4.1 and 5.4.4 are equivalent to*

$$\mathcal{E}(\mathbf{D}_1) < \sum_{i=1}^m \beta_i.$$

Finally, the following proposition ensures that (LPR) is non-degenerate with probability one.

Proposition 5.4.7 *(LPR) is non-degenerate with probability one, under the proposed stochastic model.*

Proof: This follows directly from the fact that, for each task, the vector of cost and requirement parameters is distributed according to an absolutely continuous probability distribution \square

Hereafter, we assume that Assumptions 5.4.1 and 5.4.4 are satisfied. In the remainder of this section we show that the greedy heuristic is asymptotically feasible

and optimal for two different choices of the vector of multipliers λ . In Section 5.4.2, we consider the choice $\lambda = \lambda_n^*$, where λ_n^* represents the vector of optimal dual multipliers of the capacity constraints of (LPR) when (LPR) is feasible and an arbitrary nonnegative vector when (LPR) is infeasible. (Clearly, if (LPR) is infeasible, so is the GAP.) Observe that the capacity constraints have been reformulated as \geq -constraints, so that λ_n^* is a nonnegative vector. Note that this choice depends on the problem instance. Therefore, in Section 5.4.3 we give conditions under which the sequence of random vectors $\{\lambda_n^*\}$ converges with probability one to a vector $\Lambda^* \in \mathbb{R}_+^m$, only depending on the probabilistic model. Hence, the choice $\lambda = \Lambda^*$ will be equal for all problem instances (and problem sizes, as measured by the number of tasks) corresponding to that model. Again, asymptotic feasibility and optimality will be shown for $\lambda = \Lambda^*$.

In the remainder of this section, let \mathbf{X}_n^G denote the random vector representing the solution for the GAP given by the greedy heuristic, and \mathbf{Z}_n^G be its objective value. Note that \mathbf{X}_n^G and \mathbf{Z}_n^G depend on the choice of λ . This dependence will be suppressed for notational convenience, but at any time the particular value of λ considered will be clear from the context.

5.4.2 The optimal dual multipliers

In this section we choose the vector of optimal dual multipliers of the capacity constraints of (LPR), say $\lambda_n^* \in \mathbb{R}_+^m$, as the multipliers to use in the greedy heuristic. (As mentioned above, if (LPR) is infeasible we let λ_n^* be any nonnegative vector.)

In Theorem 5.4.9, we show that the greedy heuristic is asymptotically feasible with probability one. This proof combines the results of Theorem 5.4.5 and Theorem 5.4.8 which shows that the (partial) solution found by the greedy heuristic using $\lambda = \lambda_n^*$ and the optimal solution for (LPR) coincide for almost all tasks that are feasible in the latter.

Let N_n be the set of assignments which do not coincide in x^G and in x^{LPR} , i.e.,

$$N_n = \{j = 1, \dots, n : \exists i = 1, \dots, m \text{ such that } x_{ij}^G \neq x_{ij}^{\text{LPR}}\}.$$

Basically, we will prove that the cardinality of set N_n is bounded by an expression independent of n , and thus the solutions (and, more importantly, their values) stay close when n grows.

Theorem 5.4.8 *There exists a constant R , independent of n , so that $|N_n| \leq R$ for all instances of (LPR) that are feasible and non-degenerate.*

Proof: Clearly, it would be possible to fix all feasible assignments from x^{LPR} without violating any capacity constraint. Proposition 3.5.2 ensures that the most desirable agent for each task that is feasibly assigned in x^{LPR} is equal to the agent to which it is assigned in x^{LPR} . Moreover, the same proposition shows that the initial desirabilities are such that the greedy heuristic starts by assigning tasks that are feasibly assigned in x^{LPR} . Now suppose that the greedy heuristic would reproduce all the assignments that are feasible in x^{LPR} . Then, because the remaining assignments in x^{LPR} are

infeasible with respect to the integrality constraints, x^G and x^{LPR} would differ only in those last ones. We know that then $|N_n| = |B| \leq m$ (see Benders and Van Nunen [14], and Lemma 3.5.1), and the result follows. So in the remainder of the proof we will assume that x^G and x^{LPR} differ in at least one assignment that is feasible in the latter.

While the greedy heuristic is assigning tasks that are feasibly assigned in x^{LPR} it may at some point start updating the desirabilities of the assignments still to be made due to the decreasing remaining capacities. This may cause the greedy heuristic to deviate from one of the feasible assignments in x^{LPR} . Such an assignment could use some capacity (at most \bar{A}) that x^{LPR} uses for other (feasible) assignments. Since the agent that is involved in this assignment may now not be able to accommodate all tasks that were feasibly assigned to it in x^{LPR} , other deviations from the feasible assignments in x^{LPR} will occur. However, the number of additional deviations is at most equal to $\lceil \bar{A}/\underline{A} \rceil$. In the remainder of this proof we will show that the total number of deviations is bounded by a constant independent of n . In order to make this precise, we will first bound the number of times that the desirabilities ρ must be recalculated, and then bound the number of deviations from x^{LPR} between these recalculations.

As mentioned above, we will first bound the number of times that the desirabilities ρ must be recalculated. As discussed in the geometric interpretation of the greedy heuristic (see Section 5.2), the number of times that the desirabilities ρ must be recalculated is no more than

$$m \left\lceil \frac{\max_{i=1,\dots,m; j=1,\dots,n} a_{ij}}{\min_{i=1,\dots,m; j=1,\dots,n} \{a_{ij} : a_{ij} > 0\}} \right\rceil \leq m \lceil \bar{A}/\underline{A} \rceil.$$

Now let $\ell^{(k)}$ be the iteration that induces the k -th recalculation of the values of the desirabilities ρ , and assume that this recalculation has taken place. Let M^k be the set of tasks that have been assigned in the first $\ell^{(k)}$ iterations and do not coincide with x^{LPR} . Let U^k be the set of tasks that have not been assigned in the first $\ell^{(k)}$ iterations and for which we would get a different assignment than in x^{LPR} by assigning them to their current most desirable agent (thus, if $j \in U^k$ then $x_{ij}^{\text{LPR}} \neq 1$). In other words, U^k contains the tasks that have not been assigned in the first $\ell^{(k)}$ iterations, and that would belong to N_n if they were assigned to their most desirable agent.

First note that Proposition 3.5.2 ensures that initially the most desirable agent in our greedy heuristic for each $j \notin B$ coincides with the corresponding assignment in x^{LPR} . Moreover, in the original ordering of the desirabilities, we first encounter all tasks not in B , followed by all tasks in B . Since x^G and x^{LPR} do not coincide for at least one task that is feasibly assigned in x^{LPR} , $|M^1| = 0$ and the set of tasks not assigned in the first $\ell^{(1)}$ iterations for which the most desirable agent does not coincide with the corresponding assignment in x^{LPR} is a subset of the set of infeasible assignments in x^{LPR} , thus

$$|U^1| \leq |B| \leq m.$$

It is easy to see that, for $k \geq 1$, the number of tasks that have been assigned in the first $\ell^{(k+1)}$ iterations and do not coincide with x^{LPR} is at most equal to the number

of tasks that have been assigned in the first $\ell^{(k)}$ iterations and do not coincide with x^{LPR} , plus the number of tasks that would be assigned to an agent not coinciding with x^{LPR} if they were assigned in one of the iterations $\ell^{(k)} + 1, \dots, \ell^{(k+1)}$. In other words,

$$|M^{k+1}| \leq |M^k| + |U^k|. \quad (5.3)$$

Moreover, the assignments made in the last $\ell^{(k+1)} - \ell^{(k)}$ iterations that were different from the corresponding assignment in x^{LPR} could cause additional deviations from x^{LPR} . In particular, each of these assignments could cause at most $\lceil \bar{A}/\underline{A} \rceil$ assignments still to be made to deviate from x^{LPR} . Thus,

$$\begin{aligned} |U^{k+1}| &\leq |U^k| + (|M^{k+1}| - |M^k|) \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \\ &\leq |U^k| + |M^{k+1}| \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \\ &\leq |U^k| + (|M^k| + |U^k|) \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \quad \text{using inequality (5.3)} \\ &\leq (|M^k| + |U^k|) \left(1 + \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \right). \end{aligned}$$

Using the hypothesis of induction, it can now be shown that

$$\begin{aligned} |M^k| &\leq m \left(2 + \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \right)^{k-2} \\ |U^k| &\leq m \left(2 + \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \right)^{k-1} \end{aligned}$$

for each k .

If the number of times the desirabilities are recalculated is equal to k^* , then $N_n \subseteq M^{k^*} \cup U^{k^*}$, and thus

$$|N_n| \leq |M^{k^*+1}| \leq m \left(2 + \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \right)^{k^*-1}.$$

The final result now follows by observing that $k^* \leq m \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil$. □

Now we are able to prove asymptotic feasibility and optimality of the greedy heuristic with the pseudo-cost function $f_{\lambda_n^*}$.

Theorem 5.4.9 *The greedy heuristic for $\lambda = \lambda_n^*$ is asymptotically feasible with probability one.*

Proof: By definition of the set N_n we have that

$$x_{ij}^G = x_{ij}^{\text{LPR}} \quad \text{for all } j \notin N_n; i = 1, \dots, m.$$

Let \mathbf{N}_n denote the random set for which N_n is its realization. Then,

$$\begin{aligned}
 \frac{1}{n} \left(\sum_{i=1}^m b_i - \sum_{i=1}^m \sum_{j \notin \mathbf{N}_n} \mathbf{A}_{ij} \mathbf{X}_{ij}^G \right) &= \sum_{i=1}^m \beta_i - \frac{1}{n} \sum_{i=1}^m \sum_{j \notin \mathbf{N}_n} \mathbf{A}_{ij} \mathbf{X}_{ij}^G \\
 &= \sum_{i=1}^m \beta_i - \frac{1}{n} \sum_{i=1}^m \sum_{j \notin \mathbf{N}_n} \mathbf{A}_{ij} \mathbf{X}_{ij}^{\text{LPR}} \\
 &\geq \sum_{i=1}^m \beta_i - \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n \mathbf{A}_{ij} \mathbf{X}_{ij}^{\text{LPR}} \\
 &> 0
 \end{aligned} \tag{5.4}$$

with probability one as n goes to ∞ , where inequality (5.4) follows from Theorem 5.4.5. To assign the remaining tasks it suffices to show that as n goes to infinity

$$\sum_{i=1}^m \left\lfloor \frac{b_i - \sum_{j=1}^n \mathbf{A}_{ij} \mathbf{X}_{ij}^{\text{LPR}}}{\bar{A}} \right\rfloor \geq |\mathbf{N}_n|$$

which is true if

$$\sum_{i=1}^m \left(\frac{b_i - \sum_{j=1}^n \mathbf{A}_{ij} \mathbf{X}_{ij}^{\text{LPR}}}{\bar{A}} \right) \geq m + |\mathbf{N}_n|$$

or

$$\frac{1}{n} \sum_{i=1}^m \left(b_i - \sum_{j=1}^n \mathbf{A}_{ij} \mathbf{X}_{ij}^{\text{LPR}} \right) \geq \frac{1}{n} (m + |\mathbf{N}_n|) \bar{A}.$$

From inequality (5.4), it is enough to prove that

$$\frac{1}{n} (m + |\mathbf{N}_n|) \bar{A} \rightarrow 0$$

with probability one as n goes to infinity, which follows from Theorem 5.4.8. \square

In Theorem 5.4.10, we show that the greedy heuristic for $\lambda = \lambda_n^*$ is asymptotically optimal with probability one. The proof is similar to the proof of Theorem 5.4.9.

Theorem 5.4.10 *The greedy heuristic for $\lambda = \lambda_n^*$ is asymptotically optimal with probability one.*

Proof: From Theorem 5.4.9 we know that the greedy heuristic for $\lambda = \lambda_n^*$ is asymptotically feasible with probability one. It thus suffices to show that

$$\left(\frac{1}{n} \mathbf{Z}_n^G - \frac{1}{n} \mathbf{Z}_n^{\text{LPR}} \right) \rightarrow 0$$

with probability one when n goes to infinity. By definition, we have that

$$\begin{aligned}
 \frac{1}{n} \mathbf{Z}_n^G - \frac{1}{n} \mathbf{Z}_n^{\text{LPR}} &= \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n \mathbf{C}_{ij} \mathbf{X}_{ij}^G - \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n \mathbf{C}_{ij} \mathbf{X}_{ij}^{\text{LPR}} \\
 &= \frac{1}{n} \sum_{i=1}^m \sum_{j \in \mathbf{N}_n} \mathbf{C}_{ij} \mathbf{X}_{ij}^G - \frac{1}{n} \sum_{i=1}^m \sum_{j \in \mathbf{N}_n} \mathbf{C}_{ij} \mathbf{X}_{ij}^{\text{LPR}} \\
 &\leq \overline{C} \frac{1}{n} \sum_{i=1}^m \sum_{j \in \mathbf{N}_n} \mathbf{X}_{ij}^G - \underline{C} \frac{1}{n} \sum_{i=1}^m \sum_{j \in \mathbf{N}_n} \mathbf{X}_{ij}^{\text{LPR}} \\
 &= (\overline{C} - \underline{C}) \cdot \frac{|\mathbf{N}_n|}{n}.
 \end{aligned}$$

The result then follows from Theorem 5.4.8. \square

The asymptotic optimality of the greedy heuristic has been proved for $\lambda = \lambda_n^*$. However, using this choice the vector of multipliers depends on the problem instance. In the following section, we will derive conditions under which a single vector of multipliers Λ^* suffices for all problem instances and problem sizes (as measured by the number of tasks) under a given probabilistic model. (See Rinnooy Kan, Stougie and Vercellis [108] for an analogous result for a class of generalized greedy heuristics for the Multi-Knapsack Problem.)

5.4.3 A unique vector of multipliers

Finding the multipliers

In this section we will find a candidate for Λ^* so that the greedy heuristic with the pseudo-cost function f_{Λ^*} is asymptotically feasible and optimal with probability one.

Let $L : \mathbb{R}^m \rightarrow \mathbb{R}$ be the real-valued function defined as

$$L(\lambda) = \mathcal{E} \left(\min_{i=1, \dots, m} (\mathbf{C}_{i1} + \lambda_i \mathbf{A}_{i1}) \right) - \lambda^\top \beta. \quad (5.5)$$

Recall from Theorem 5.4.2 that the maximum value of the function L on the set \mathbb{R}_+^m is equal to θ . In this section, we will show that the function L has a unique maximizer, say Λ^* , over the nonnegative orthant.

First we define a sequence of functions L_n which converges to L . This allows us to prove some properties of the function L . Let $L_n : \mathbb{R}^m \rightarrow \mathbb{R}$ be the real-valued function defined as

$$L_n(\lambda) = \frac{1}{n} \sum_{j=1}^n \min_{i=1, \dots, m} (c_{ij} + \lambda_i a_{ij}) - \lambda^\top \beta.$$

Recall that $\lambda = \lambda_n^*$ is defined as the vector of optimal dual multipliers of the capacity constraints of (LPR) when (LPR) is feasible and an arbitrary nonnegative vector when (LPR) is infeasible.

Proposition 5.4.11 *The function L_n satisfies the following properties:*

- (i) *If (LPR) is feasible, then λ_n^* is the maximizer of function L_n on the set of nonnegative vectors $\lambda \in \mathbb{R}_+^m$.*
- (ii) *$L_n(\lambda_n^*) \rightarrow \theta$ with probability one when n goes to infinity.*
- (iii) *For n large enough, λ_n^* has at least one component equal to zero with probability one.*

Proof: Let $(\lambda, v) \in \mathbb{R}_+^m \times \mathbb{R}^n$ be a feasible solution for (D). In the proof of Proposition 3.5.2, we deduced that without loss of optimality that

$$v_j = \min_{i=1, \dots, m} (c_{ij} + \lambda_i a_{ij}).$$

Therefore, the optimal value of (D) can be written as

$$\begin{aligned} & \max_{\lambda \geq 0} \left(\sum_{j=1}^n \min_{i=1, \dots, m} (c_{ij} + \lambda_i a_{ij}) - \lambda^\top b \right) = \\ &= n \max_{\lambda \geq 0} \left(\frac{1}{n} \sum_{j=1}^n \min_{i=1, \dots, m} (c_{ij} + \lambda_i a_{ij}) - \lambda^\top \beta \right) \\ &= n \max_{\lambda \geq 0} L_n(\lambda), \end{aligned}$$

and then Claim (i) follows.

By strong duality, Assumption 5.4.1 and Proposition 5.4.7, we have that $\frac{1}{n} \mathbf{Z}_n^{\text{LPR}} = L_n(\lambda_n^*)$. Claim (ii) now follows by using Theorem 5.4.2.

In the proof of Theorem 5.4.3, Romeijn and Piersma [110] define the function $\psi_n : \mathbb{R}_+ \rightarrow \mathbb{R}$ as

$$\begin{aligned} \psi_n(x) &= \min_{\lambda \geq x\mathbf{e}} \left(\lambda^\top \beta - \frac{1}{n} \sum_{j=1}^n \min_{i=1, \dots, m} (c_{ij} + \lambda_i a_{ij}) \right) \\ &= - \max_{\lambda \geq x\mathbf{e}} L_n(\lambda). \end{aligned}$$

In that proof it is shown that the sequence $\{\psi_n\}$ converges pointwise to the function ψ defined in Theorem 5.4.3. Moreover, under Assumption 5.4.4, it is deduced that

$$\liminf_{n \rightarrow \infty} (\psi_n)_+'(0) > 0 \quad (5.6)$$

with probability one. In particular, $\psi_n(0) = -\max_{\lambda \in \mathbb{R}_+^m} L_n(\lambda)$. From inequality (5.6), eventually, $\psi_n(\varepsilon) \geq \psi_n(0)$ (where $\varepsilon > 0$). Thus, the maximum of the function L_n on \mathbb{R}_+^m cannot be reached in a vector with all components strictly positive. Thus Claim (iii) follows. \square

Now we are able to prove some properties of the function L .

Lemma 5.4.12 *The function L satisfies the following properties:*

- (i) *The function L is concave.*
- (ii) *$L(\lambda_n^*) \rightarrow \theta$ with probability one when n goes to infinity.*

Proof: Using the Strong Law of Large Numbers, it is easy to see that the sequence of the functions L_n converges pointwise to the function L with probability one. Each of the functions L_n is concave on \mathbb{R}_+^m , since it is expressed as the algebraic sum of a linear function and the minimum of linear functions. Thus, Claim (i) follows by using pointwise convergence of L_n to L on \mathbb{R}_+^m , see Rockafellar [109].

To prove Claim (ii), we first show uniform convergence of the functions L_n to L on a compact set containing the maximizers of the functions L_n and L . Let K be the compact set on \mathbb{R}_+^m defined as

$$K = \left\{ \lambda \in \mathbb{R}^m : \lambda \geq 0, \mathcal{E} \left(\max_{s=1,\dots,m} C_{s1} - \min_{i=1,\dots,m} C_{i1} \right) - \lambda^\top \beta \geq 0 \right\}. \quad (5.7)$$

Using the Strong Law of Large Numbers, we have that

$$\begin{aligned} \Pr \left(\exists n_1 : \forall n \geq n_1, \frac{1}{n} \sum_{j=1}^n \left(\max_{s=1,\dots,m} c_{sj} - \min_{i=1,\dots,m} c_{ij} \right) \leq \right. \\ \left. \leq 1 + \mathcal{E} \left(\max_{s=1,\dots,m} C_{s1} - \min_{i=1,\dots,m} C_{i1} \right) \right) = 1. \end{aligned} \quad (5.8)$$

Proposition 5.4.11(iii) assures that if n is large enough L_n reaches its maximum in a vector with at least one component equal to zero with probability one. By increasing n_1 in (5.8) if necessary, we can assume that for each $n \geq n_1$, λ_n^* has at least one component equal to zero with probability one. We will show that, for a fixed $n \geq n_1$, each vector $\lambda \in \mathbb{R}_+^m$, with $\lambda \not\geq 0$ and $\lambda \notin K$ is no better than the origin, that is, $L_n(\lambda) \leq L_n(0)$. We have that

$$\begin{aligned} L_n(\lambda) &= \frac{1}{n} \sum_{j=1}^n \min_{i=1,\dots,m} (c_{ij} + \lambda_i a_{ij}) - \lambda^\top \beta \\ &\leq \frac{1}{n} \sum_{j=1}^n \max_{i=1,\dots,m} c_{ij} - \lambda^\top \beta \end{aligned} \quad (5.9)$$

$$\begin{aligned} &< \frac{1}{n} \sum_{j=1}^n \min_{i=1,\dots,m} c_{ij} \\ &= L_n(0) \end{aligned} \quad (5.10)$$

where inequality (5.9) follows from the fact that L_n reaches its maximum in a vector with at least one component equal to zero, and strict inequality (5.10) follows since $\lambda \notin K$ and $\lambda \in \mathbb{R}_+^m$. This means that, for each $\lambda \notin K$ so that $\lambda \in \mathbb{R}_+^m$, we have that

$$\Pr (\exists n_1 : \forall n \geq n_1, L_n(\lambda) \leq L_n(0)) = 1.$$

Since the origin belongs to K , this implies that $\lambda_n^* \in K$ for each $n \geq n_1$ with probability one. In a similar fashion we can prove that each maximizer of the function L belongs to K . Note that the set K is compact since

$$\mathcal{E} \left(\max_{s=1,\dots,m} C_{s1} - \min_{i=1,\dots,m} C_{i1} \right) < \infty.$$

Theorem 10.8 in Rockafellar [109] shows that L_n converges uniformly to L with probability one on K . Now consider the following inequality

$$|L(\lambda_n^*) - \theta| \leq |L(\lambda_n^*) - L_n(\lambda_n^*)| + |L_n(\lambda_n^*) - \theta|.$$

From the uniform convergence the first term of the right-hand side tends to zero and from Proposition 5.4.11(ii) the second term also tends to zero, and then Claim (ii) follows. \square

To prove Theorem 5.4.14, we first derive the expression of the Hessian of the function L . Before we do this, we introduce some simplifying notation. Given $c \in \mathbb{R}^m$, then we define $c_{(k)} = (c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_m) \in \mathbb{R}^{m-1}$. Moreover, we interpret $(c_{(k)}, z)$ to be equivalent to $(c_1, \dots, c_{k-1}, z, c_{k+1}, \dots, c_m)$, where the usage of either notation is dictated by convenience, and where the meaning should be clear from the context. Similarly, we define $c_{(k,i)}$ to be the vector in \mathbb{R}^{m-2} which can be obtained from c by removing both c_k and c_i .

In the next result we will suppress, for notational convenience, the index 1 in the vector (C_1, A_1) .

Lemma 5.4.13 *Let (C, A) be a random vector with absolutely continuous distribution in $[\underline{C}, \overline{C}]^m \times [\underline{A}, \overline{A}]^m$. Then the function L is twice differentiable, and the first and second order partial derivatives read as follows*

$$\begin{aligned} \frac{\partial L(\lambda)}{\partial \lambda_k} &= \mathcal{E}(A_k X_k(\lambda)) - \beta_k \\ \frac{\partial^2 L(\lambda)}{\partial \lambda_i \partial \lambda_k} &= \begin{cases} \mathcal{E}_A \left(A_k A_i \int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\overline{C}} X_{ki}(\lambda) \right. \\ \quad \left. g|_A(c_{(k)}, \min_{s \neq k} (c_s + \lambda_s A_s) - \lambda_k A_k) \, dc_{(k)} \right) & \text{if } i \neq k \\ \mathcal{E}_A \left(-A_k^2 \int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\overline{C}} \right. \\ \quad \left. g|_A(c_{(k)}, \min_{s \neq k} (c_s + \lambda_s A_s) - \lambda_k A_k) \, dc_{(k)} \right) & \text{if } i = k \end{cases} \end{aligned}$$

for each k and $i = 1, \dots, m$, where $X_k(\lambda)$ is the Boolean random variable equal to 1 if $k = \arg \min_{s=1,\dots,m} (C_s + \lambda_s A_s)$ and 0 otherwise, $X_{ki}(\lambda)$ is the Boolean random variable taking the value 1 if $i = \arg \min_{s=1,\dots,m; s \neq k} (C_s + \lambda_s A_s)$ and 0 otherwise, and $g|_A$ is the density function of the vector C conditional upon A .

Proof: For notational simplicity, we write $L(\lambda) = \tilde{L}(\lambda) - \lambda^\top \beta$ where

$$\tilde{L}(\lambda) = \mathcal{E} \left(\min_{i=1,\dots,m} (C_i + \lambda_i A_i) \right).$$

It thus suffices to calculate the first and the second order partial derivatives of \tilde{L} . The function \tilde{L} can be written as

$$\tilde{L}(\lambda) = \mathcal{E}_{\mathbf{A}} \left(\sum_{i=1}^m \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \int_{\underline{\mathcal{C}}}^{\min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_i \mathbf{A}_i} (c_i + \lambda_i \mathbf{A}_i) g(c) \, dc_i \, dc_{(i)} \right)$$

where g is the density function of vector \mathbf{C} . Here we have assumed without loss of generality that the vectors \mathbf{C} and \mathbf{A} are independent. If they are not, then the density function g should be replaced by $g|_{\mathbf{A}}$, the density function of \mathbf{C} conditioned by \mathbf{A} , throughout this proof.

By the Dominated Convergence Theorem, the first order partial derivative of \tilde{L} with respect to λ_k , for each $k = 1, \dots, m$, is equal to

$$\begin{aligned} \frac{\partial \tilde{L}(\lambda)}{\partial \lambda_k} &= \\ &= \frac{\partial}{\partial \lambda_k} \left(\mathcal{E}_{\mathbf{A}} \left[\int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \int_{\underline{\mathcal{C}}}^{\min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k} (c_k + \lambda_k \mathbf{A}_k) \right. \right. \\ &\quad \left. \left. g(c) \, dc_k \, dc_{(k)} \right] + \right. \\ &\quad \left. + \mathcal{E}_{\mathbf{A}} \left[\sum_{i \neq k} \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \int_{\underline{\mathcal{C}}}^{\min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_i \mathbf{A}_i} (c_i + \lambda_i \mathbf{A}_i) \right. \right. \\ &\quad \left. \left. g(c) \, dc_i \, dc_{(i)} \right] \right) \\ &= \mathcal{E}_{\mathbf{A}} \left[\int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \int_{\underline{\mathcal{C}}}^{\min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k} \mathbf{A}_k g(c) \, dc_k \, dc_{(k)} \right] \\ &\quad - \mathcal{E}_{\mathbf{A}} \left[\int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \mathbf{A}_k \min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) \right. \\ &\quad \left. g \left(c_{(k)}, \min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k \right) \, dc_{(k)} \right] \end{aligned} \quad (5.11)$$

$$\begin{aligned} &+ \sum_{i \neq k} \mathcal{E}_{\mathbf{A}} \left[\int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) \frac{\partial}{\partial \lambda_k} \left(\min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) \right) \right. \\ &\quad \left. g \left(c_{(i)}, \min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_i \mathbf{A}_i \right) \, dc_{(i)} \right]. \end{aligned} \quad (5.12)$$

We will show that the terms (5.11) and (5.12) are equal, and thus their difference vanishes. We observe that (5.11) can be written as follows

$$\begin{aligned} &\mathcal{E}_{\mathbf{A}} \left[\int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \cdots \int_{\underline{\mathcal{C}}}^{\overline{\mathcal{C}}} \mathbf{A}_k \min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) \right. \\ &\quad \left. g \left(c_{(k)}, \min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k \right) \, dc_{(k)} \right] = \end{aligned}$$

$$\begin{aligned}
&= \mathcal{E}_{\mathbf{A}} \left[\sum_{i \neq k} \int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\min_{s \neq k, i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_i \mathbf{A}_i} \mathbf{A}_k (c_i + \lambda_i \mathbf{A}_i) \right. \\
&\quad \left. g(c_{(k)}, c_i + \lambda_i \mathbf{A}_i - \lambda_k \mathbf{A}_k) \, dc_i \, dc_{(k, i)} \right] \\
&= \mathcal{E}_{\mathbf{A}} \left[\sum_{i \neq k} \int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C} + \lambda_i \mathbf{A}_i - \lambda_k \mathbf{A}_k}^{\min_{s \neq k, i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k} \mathbf{A}_k (c_k + \lambda_k \mathbf{A}_k) \right. \\
&\quad \left. g(c_{(i)}, c_k + \lambda_k \mathbf{A}_k - \lambda_i \mathbf{A}_i) \, dc_k \, dc_{(i, k)} \right].
\end{aligned}$$

The first equality has been obtained by varying the index i where $\min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s)$ is reached, and the second one by making a change of variables. With respect to (5.12), the partial derivative $\frac{\partial}{\partial \lambda_k} (\min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s))$ has value different from zero only when $\min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s)$ is reached at $s = k$. Thus, we have that

$$\begin{aligned}
&\sum_{i \neq k} \mathcal{E}_{\mathbf{A}} \left[\int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\overline{C}} \min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) \frac{\partial}{\partial \lambda_k} \left(\min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) \right) \right. \\
&\quad \left. g \left(c_{(i)}, \min_{s \neq i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_i \mathbf{A}_i \right) \, dc_{(i)} \right] = \\
&= \mathcal{E}_{\mathbf{A}} \left[\sum_{i \neq k} \int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\min_{s \neq k, i} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k} \mathbf{A}_k (c_k + \lambda_k \mathbf{A}_k) \right. \\
&\quad \left. g(c_{(i)}, c_k + \lambda_k \mathbf{A}_k - \lambda_i \mathbf{A}_i) \, dc_k \, dc_{(i, k)} \right].
\end{aligned}$$

Thus (5.12) – (5.11) can be written as

$$\begin{aligned}
&\mathcal{E}_{\mathbf{A}} \left[\sum_{i \neq k} \int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\underline{C} + \lambda_i \mathbf{A}_i - \lambda_k \mathbf{A}_k} \mathbf{A}_k (c_k + \lambda_k \mathbf{A}_k) \right. \\
&\quad \left. g(c_{(i)}, c_k + \lambda_k \mathbf{A}_k - \lambda_i \mathbf{A}_i) \, dc_k \, dc_{(i, k)} \right]. \tag{5.13}
\end{aligned}$$

Now note that, for all $c_k \in [\underline{C}, \underline{C} + \lambda_i a_i - \lambda_k a_k]$, we have that $c_k + \lambda_k a_k - \lambda_i a_i \leq \underline{C}$, so that expression (5.13) equals 0. Thus the first partial derivatives of \tilde{L} can be written as

$$\frac{\partial \tilde{L}(\lambda)}{\partial \lambda_k} = \mathcal{E}_{\mathbf{A}} \left[\int_{\underline{C}}^{\overline{C}} \cdots \int_{\underline{C}}^{\overline{C}} \int_{\underline{C}}^{\min_{s \neq k} (c_s + \lambda_s \mathbf{A}_s) - \lambda_k \mathbf{A}_k} \mathbf{A}_k g(c) \, dc \right] = \mathcal{E}(\mathbf{A}_k \mathbf{X}_k(\lambda)).$$

In a similar way, we can derive the expression of the second order partial derivatives of the function L . \square

We are now able to show the first main result of this section.

Theorem 5.4.14 *If the density of $(\mathbf{C}_1, \mathbf{A}_1)$ is strictly positive over a convex open set, then L has a unique maximizer on the set \mathbb{R}_+^m .*

Proof: For notational convenience, we again suppress the index 1 in the vector $(\mathbf{C}_1, \mathbf{A}_1)$. From the proof of Lemma 5.4.12, we know that

$$\sup_{\lambda \in \mathbb{R}_+^m} L(\lambda) = \max_{\lambda \in K} L(\lambda)$$

where K is the compact set defined by (5.7). Thus, the function L has at least one maximizer Λ^* on \mathbb{R}_+^m . In the following we will show uniqueness of this maximizer.

Denote by I the set of non-active capacity constraints for Λ^* with dual multiplier equal to zero, that is

$$I = \{i = 1, \dots, m : \Lambda_i^* = 0, \mathcal{E}(\mathbf{A}_i \mathbf{X}_i(\Lambda^*)) < \beta_i\}.$$

From the sufficient second order condition, it is enough to show that $H(\Lambda^*)$, the Hessian of the function L at Λ^* , is negative definite on the subspace

$$M = \{y \in \mathbb{R}^m : y_\ell = 0, \text{ for each } \ell \in I\}.$$

Now let $y \in M$, $y \neq 0$, and evaluate the quadratic form associated with the Hessian of the function L in Λ^* :

$$\begin{aligned} y^\top H(\Lambda^*) y &= \\ &= \sum_{k, i \notin I; i > k} 2y_k y_i \mathcal{E}_{\mathbf{A}} \left[\mathbf{A}_k \mathbf{A}_i \int_{\underline{C}}^{\overline{C}} \dots \int_{\underline{C}}^{\overline{C}} \mathbf{X}_{ki}(\Lambda^*) \right. \\ &\quad \left. g|_{\mathbf{A}} \left(c_{(k)}, \min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* \mathbf{A}_s) - \Lambda_k^* \mathbf{A}_k \right) dc_{(k)} \right] \\ &\quad + \sum_{k \notin I} y_k^2 \mathcal{E}_{\mathbf{A}} \left[-\mathbf{A}_k^2 \int_{\underline{C}}^{\overline{C}} \dots \int_{\underline{C}}^{\overline{C}} \right. \\ &\quad \left. g|_{\mathbf{A}} \left(c_{(k)}, \min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* \mathbf{A}_s) - \Lambda_k^* \mathbf{A}_k \right) dc_{(k)} \right] \\ &= - \sum_{k, i \notin I; i > k} \mathcal{E}_{\mathbf{A}} \left[(y_k \mathbf{A}_k - y_i \mathbf{A}_i)^2 \int_{\underline{C}}^{\overline{C}} \dots \int_{\underline{C}}^{\overline{C}} \mathbf{X}_{ki}(\Lambda^*) \right. \\ &\quad \left. g|_{\mathbf{A}} \left(c_{(k)}, \min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* \mathbf{A}_s) - \Lambda_k^* \mathbf{A}_k \right) dc_{(k)} \right] \\ &\quad - \sum_{k \notin I} y_k^2 \sum_{\ell \in I} \mathcal{E}_{\mathbf{A}} \left[\mathbf{A}_k^2 \int_{\underline{C}}^{\overline{C}} \dots \int_{\underline{C}}^{\overline{C}} \mathbf{X}_{k\ell}(\Lambda^*) \right. \\ &\quad \left. g|_{\mathbf{A}} \left(c_{(k)}, \min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* \mathbf{A}_s) - \Lambda_k^* \mathbf{A}_k \right) dc_{(k)} \right]. \end{aligned}$$

Since the vector (\mathbf{C}, \mathbf{A}) has positive density on an open set, so does \mathbf{A} , and then

$$\mathcal{E}_{\mathbf{A}}[(y_k \mathbf{A}_k - y_i \mathbf{A}_i)^2] > 0 \quad \text{if } (y_k, y_i) \neq (0, 0).$$

To prove that $y^\top H(\Lambda^*)y > 0$, it is enough to show that for each $k \notin I$ there exists a vector $(c_{(k)}, a)$ such that

$$\mu_k(c_{(k)}, a) + \Lambda_k^* a_k < \min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* a_s) < \nu_k(c_{(k)}, a) + \Lambda_k^* a_k$$

where $(\mu_k(c_{(k)}, a), \nu_k(c_{(k)}, a))$ is the interval in which \mathbf{C}_k has positive density when $(\mathbf{C}_{(k)}, \mathbf{A}) = (c_{(k)}, a)$.

First suppose that $\min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* a_s) < \mu_k(c_{(k)}, a) + \Lambda_k^* a_k$ for all vectors $(c_{(k)}, a)$ with positive density. Then, there exists a neighborhood of Λ^* so that

$$\min_{s=1, \dots, m; s \neq k} (c_s + \lambda_s a_s) < \mu_k(c_{(k)}, a) + \lambda_k a_k.$$

Therefore, for each λ in this neighborhood, we have that

$$\begin{aligned} L(\lambda) &= \mathcal{E} \left(\min_{i=1, \dots, m} (\mathbf{C}_i + \lambda_i \mathbf{A}_i) \right) - \lambda^\top \beta \\ &= \mathcal{E} \left(\min_{i=1, \dots, m; i \neq k} (\mathbf{C}_i + \lambda_i \mathbf{A}_i) \right) - \lambda^\top \beta. \end{aligned}$$

Since $k \notin I$, $\Lambda_k^* > 0$ and we can decrease it so that we obtain a new vector where L has smaller value. (Recall that $\beta_k > 0$.) But this contradicts the fact that Λ^* is a maximizer of L .

Similarly, if $\min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* a_s) > \nu_k(c_{(k)}, a) + \Lambda_k^* a_k$, there exists a neighborhood of Λ^* so that

$$L(\lambda) = \mathcal{E} \left(\min_{i=1, \dots, m} (\mathbf{C}_i + \lambda_i \mathbf{A}_i) \right) - \lambda^\top \beta = \mathcal{E}(\mathbf{C}_k + \lambda_k \mathbf{A}_k) - \lambda^\top \beta.$$

Therefore, we can decrease Λ_k^* so that we obtain a new vector where L has smaller value. (Recall that $\Lambda_k^* > 0$ and $\mathcal{E}(\mathbf{A}_k) > \beta_k$.) But this contradicts again the fact that Λ^* is a maximizer of L .

Then, there exists a vector $(c_{(k)}, a)$ so that

$$\nu_k(c_{(k)}, a) + \Lambda_k^* a_k \leq \min_{s=1, \dots, m; s \neq k} (c_s + \Lambda_s^* a_s) \leq \mu_k(c_{(k)}, a) + \Lambda_k^* a_k,$$

and the result follows by observing that (\mathbf{C}, \mathbf{A}) is strictly positive over a convex open set. \square

Proposition 5.4.15 *If the density of $(\mathbf{C}_1, \mathbf{A}_1)$ is strictly positive on a convex open set, there exists a unique vector $\Lambda^* \in \mathbb{R}_+^m$ such that*

$$\lambda_n^* \rightarrow \Lambda^*$$

with probability one when n goes to infinity.

Proof: This result follows immediately by using Corollary 27.2.2 in Rockafellar [109], Lemma 5.4.12, Theorem 5.4.14, and the remark following equation (5.5) at the beginning of this section. \square

In the following section we prove that the greedy heuristic with the pseudo-cost function f_{Λ^*} is asymptotically feasible and optimal with probability one.

Proving asymptotic optimality

To prove asymptotic feasibility and optimality of the greedy heuristic with the pseudo-cost function f_{Λ^*} , we will show that the (partial) solution found by the greedy heuristic and the optimal solution for (LPR) coincide for almost all the tasks that are feasible assigned in the latter. A similar result is stated for the greedy heuristic with the pseudo-cost function $f_{\lambda_n^*}$ in Theorem 5.4.8.

Let ρ be the initial desirabilities given by the greedy heuristic with the pseudo-cost function f_{Λ^*} . For notational simplicity, we have suppressed the dependence of ρ on Λ^* . First, we will define a barrier ε_n such that the most desirable agent for each task j with $\rho_j > \varepsilon_n$ is equal to the agent to which it is assigned in x^{LPR} . The barrier ε_n is defined as

$$\varepsilon_n = \sup_{j=1,\dots,n} \max_{i=1,\dots,m; \ell=1,\dots,m; \ell \neq i} ((\Lambda_\ell^* - (\lambda_n^*)_\ell) a_{\ell j} - (\Lambda_i^* - (\lambda_n^*)_i) a_{ij})$$

where $(\lambda_n^*)_\ell$ represents the ℓ -th component of vector $\lambda_n^* \in \mathbb{R}_+^m$. Note that $\varepsilon_n \geq 0$.

Proposition 5.4.16 *For each task j so that $\rho_j > \varepsilon_n$, the following statements hold:*

(i) *There exists $i = 1, \dots, m$ such that*

$$c_{ij} + (\lambda_n^*)_i a_{ij} = \min_{s=1,\dots,m} (c_{sj} + (\lambda_n^*)_s a_{sj}),$$

and

$$c_{ij} + (\lambda_n^*)_i a_{ij} < \min_{s=1,\dots,m; s \neq i} (c_{sj} + (\lambda_n^*)_s a_{sj}).$$

(ii) *Moreover,*

$$\arg \min_{s=1,\dots,m} (c_{sj} + \Lambda_s^* a_{sj}) = \arg \min_{s=1,\dots,m} (c_{sj} + (\lambda_n^*)_s a_{sj}).$$

Proof: Let j be a task so that $\rho_j > \varepsilon_n$. Since ε_n is nonnegative, the desirability of task j is strictly positive, and then $i_j = \arg \min_{s=1,\dots,m} (c_{sj} + \Lambda_s^* a_{sj})$ is unique. To prove Claim (ii) it is enough to show that

$$i_j = \arg \min_{s=1,\dots,m} (c_{sj} + (\lambda_n^*)_s a_{sj}).$$

Using the definition of ε_n , $\rho_j > \varepsilon_n$ implies that

$$\rho_j > \max_{i=1,\dots,m; \ell=1,\dots,m; \ell \neq i} ((\Lambda_\ell^* - (\lambda_n^*)_\ell) a_{\ell j} - (\Lambda_i^* - (\lambda_n^*)_i) a_{ij}).$$

Since $\rho_j = \min_{s=1,\dots,m; s \neq i_j} ((c_{sj} + \Lambda_s^* a_{sj}) - (c_{i_j j} + \Lambda_{i_j}^* a_{i_j j}))$, we thus have that

$$\begin{aligned} & \min_{s=1,\dots,m; s \neq i_j} \left((c_{sj} + \Lambda_s^* a_{sj}) - (c_{i_j j} + \Lambda_{i_j}^* a_{i_j j}) \right) > \\ & > \max_{\ell=1,\dots,m; \ell \neq i_j} \left((\Lambda_\ell^* - (\lambda_n^*)_\ell) a_{\ell j} - (\Lambda_{i_j}^* - (\lambda_n^*)_{i_j}) a_{i_j j} \right). \end{aligned}$$

This implies that, for each $s \neq i_j$, it holds

$$(c_{sj} + \Lambda_s^* a_{sj}) - (c_{i_j j} + \Lambda_{i_j}^* a_{i_j j}) > (\Lambda_s^* - (\lambda_n^*)_s) a_{sj} - (\Lambda_{i_j}^* - (\lambda_n^*)_{i_j}) a_{i_j j},$$

then

$$c_{sj} + (\lambda_n^*)_s a_{sj} > c_{i_j j} + (\lambda_n^*)_{i_j} a_{i_j j}$$

and Claim (ii) follows. Moreover, from the last inequality $\min_{s=1, \dots, m} (c_{sj} + (\lambda_n^*)_s a_{sj})$ is reached only when $s = i_j$, thus Claim (i) also holds. \square

From this result we can derive the following corollary.

Corollary 5.4.17 *If (LPR) is feasible and non-degenerate, each task j for which $\rho_j > \varepsilon_n$ is feasibly assigned by (LPR).*

Proof: The result follows immediately from Proposition 5.4.16(i) and Proposition 3.5.2(i). \square

We will now study the behaviour of ε_n as n goes to infinity.

Lemma 5.4.18 *ε_n tends to 0 with probability one as n goes to infinity.*

Proof: This result follows immediately from Proposition 5.4.15. \square

We now investigate how large is the set of tasks for which $\rho_j \leq \varepsilon_n$. Let R_n denote the set of tasks for which the initial desirabilities with the pseudo-cost function f_{Λ^*} is not above the barrier ε_n , i.e.,

$$R_n = \{j = 1, \dots, n : \rho_j \leq \varepsilon_n\}.$$

Proposition 5.4.19 *We have that*

$$\frac{|R_n|}{n} \rightarrow 0$$

with probability one when n goes to infinity.

Proof: Suppose that the result is not true. Since the original sequence lies completely in the compact set $[0, 1]$, there exists a subsequence $\left\{ \frac{|R_{n_k}|}{n_k} \right\}$ which tends to $\ell > 0$ with probability one.

Let F_{ρ_1} be the distribution function of the random variable ρ_1 . Observe that random variables ρ_j , $j = 1, \dots, n$, are i.i.d.. Consider the sequence of random variables \bar{Y}_j equal to 1 if $\rho_j \leq F_{\rho_1}^{-1}\left(\frac{\ell}{2}\right)$, and 0 otherwise. The variables \bar{Y}_j , $j = 1, \dots, n$, are i.i.d. as a Bernoulli random variable with parameter

$$\Pr\left(\rho_j \leq F_{\rho_1}^{-1}\left(\frac{\ell}{2}\right)\right) = F_{\rho_1}\left(F_{\rho_1}^{-1}\left(\frac{\ell}{2}\right)\right) = \frac{\ell}{2}.$$

Using Lemma 5.4.18 and the absolute continuity of the variables \mathbf{C}_1 and \mathbf{A}_1 , there exists a constant $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\varepsilon_n < F_{\rho_1}^{-1}(\frac{\ell}{2})$, which implies that for each $n_k \geq n_0$ we have that

$$\frac{|\mathbf{R}_{n_k}|}{n_k} \leq \frac{\sum_{j=1}^{n_k} \bar{\mathbf{Y}}_j}{n_k} \rightarrow \frac{\ell}{2}$$

where the convergence follows by the Strong Law of the Large Numbers. But this contradicts the fact that $\frac{|\mathbf{R}_{n_k}|}{n_k}$ tends to ℓ with probability one. \square

Now we are able to show that the (partial) solution found by the greedy heuristic using $\lambda = \Lambda^*$ and the optimal solution for (LPR) coincide for almost all the tasks that are feasible in the latter. Recall that N_n was defined in Section 5.4.2 as the set of assignments which do not coincide in x^G and in x^{LPR} , i.e.,

$$N_n = \{j = 1, \dots, n : \exists i = 1, \dots, m \text{ such that } x_{ij}^G \neq x_{ij}^{\text{LPR}}\}.$$

Theorem 5.4.20 *Suppose that (LPR) is feasible and non-degenerate. Then, it holds*

$$|N_n| \leq |R_n| \left(2 + \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil \right)^{m \left\lceil \frac{\bar{A}}{\underline{A}} \right\rceil - 1}.$$

Proof: We will follow similar steps as in the proof of Theorem 5.4.8.

All feasible assignments from x^{LPR} can be fixed without violating any capacity constraint. Corollary 5.4.17 ensures that each task $j \notin R_n$ is feasibly assigned in x^{LPR} . Proposition 5.4.16 and Proposition 3.5.2 say that the most desirable agent for each task $j \notin R_n$ is equal to the agent to which it is assigned in x^{LPR} . Moreover, the greedy heuristic starts by assigning tasks $j \notin R_n$ and therefore tasks which are feasibly assigned in x^{LPR} . Now suppose that the greedy heuristic would reproduce all the assignments from $j \notin R_n$. Then, $|N_n| \leq |R_n|$, and the desired inequality follows. So it remains to prove the result when x^G and x^{LPR} differ in at least the assignment of a task $j \notin R_n$.

Now using the same recursion as in Theorem 5.4.8, where set B must be substituted by set R_n , we can prove the desired result. \square

Now we are able to prove the asymptotic feasibility of the greedy heuristic when $\lambda = \Lambda^*$.

Theorem 5.4.21 *The greedy heuristic for $\lambda = \Lambda^*$ is asymptotically feasible with probability one.*

Proof: Similarly as in Theorem 5.4.9, it is enough to prove that

$$\frac{1}{n}(m + |\mathbf{N}_n|) \bar{A} \rightarrow 0$$

with probability one when n goes to infinity. But this follows from Theorem 5.4.20 and Proposition 5.4.19. \square

Finally, we can prove asymptotic optimality with probability one of the greedy heuristic when $\lambda = \Lambda^*$.

Theorem 5.4.22 *The greedy heuristic for $\lambda = \Lambda^*$ is asymptotically optimal with probability one.*

Proof: The result follows similarly as Theorem 5.4.10. \square

5.5 Numerical illustrations

5.5.1 Introduction

In this chapter we have proposed the family of pseudo-cost functions

$$\{f_\lambda(i, j) = c_{ij} + \lambda_i a_{ij} : \lambda \in \mathbb{R}_+^m\}$$

for the greedy heuristic for the GAP given by Martello and Toth [88] (see Section 2.3.1). Special attention has been paid to the pseudo-cost function $f_{\lambda^*}(i, j) = c_{ij} + \lambda_i^* a_{ij}$ where λ_i^* is the optimal dual multiplier of the i -th capacity constraint of the LP-relaxation of the GAP. (For notational simplicity we have suppressed the dependence on n .)

This section contains two classes of numerical illustrations. First, we compare the performance of the pseudo-cost function f_{λ^*} against the four ones that Martello and Toth [88] proposed (see Section 5.1). Second, we analyze the quality of the solution obtained by solving the GAP with the greedy heuristic using the pseudo-cost function f_{λ^*} and improved by the two local exchange procedures described in Section 2.3.2.

5.5.2 Comparison with Martello and Toth

In this section we test the pseudo-cost function proposed in Section 5.4.2 against the four pseudo-cost functions suggested by Martello and Toth [88] (see Section 5.1).

Similarly to Chapter 4, the cost parameters \mathbf{C}_{ij} have been generated uniformly between 10 and 50, and the requirement parameters \mathbf{A}_{ij} uniformly between 5 and 25. We have chosen the capacities $b_i = b = \beta \cdot n/m$ where

$$\beta = \mu \times \frac{5m + 25}{m + 1}.$$

Asymptotic feasibility conditions in the probabilistic sense for this stochastic model were given by Romeijn and Piersma [110] and explicit expressions have been derived in Chapter 4 for requirements with increasing failure rate distribution. For these parameters, asymptotic feasibility with probability one is ensured if $\mu > 1$ (see Corollary 4.2.6). To account for the asymptotic nature of this feasibility guarantee, we have set $\mu = 1.5$ to obtain feasible problem instances for finite n . Tighter problems instances were generated, but, in this case, even the LP-relaxation of the GAP was sometimes infeasible (recall that the bound only ensures asymptotic feasibility!).

We have chosen two different values of the number of agents $m = 3$ and $m = 5$. For each of those values the number of tasks was varied from $n = 30$ until $n = 100$ in increments of 5 tasks. For each size of the problem we have generated 50 problem instances. All the runs were performed on a PC with a 350 MHz Pentium II processor and 128 MB RAM. All LP-relaxations were solved using CPLEX 6.5 [33].

Figures 5.1–5.4 illustrate the behaviour of the greedy heuristic with the pseudo-cost functions c_{ij} , a_{ij} , $(c_{ij} - 60)/a_{ij}$ and $c_{ij} + \lambda_i^* a_{ij}$. (Observe that, for all $j = 1, \dots, n$, $60 > \max_{i=1, \dots, m} c_{ij}$.) Recall that the three first choices correspond to the pseudo-cost functions (i), (ii) and (iv) proposed by Martello and Toth [88] and the last one corresponds to the pseudo-cost function analyzed in Section 5.4.2. For the particular type of problem instances that we have generated, the pseudo-cost function (iii) is equal to $a_{ij}/b_i = a_{ij}/b$. Therefore, the solution obtained by the greedy heuristic with the pseudo-cost function (iii) coincides with the one obtained using the pseudo-cost function (ii).

Figures 5.1 and 5.2 show the behaviour of the average fraction of problem instances where a feasible solution was found by the greedy heuristic. We observe that the pseudo-cost functions a_{ij} and $(c_{ij} - 60)/a_{ij}$ behave similarly. When evaluating an assignment through the requirement, a feasible solution was always found by the greedy heuristic for both values of the number of agents. The pseudo-cost function $(c_{ij} - 60)/a_{ij}$ only failed in one problem instance of size $m = 3$ and $n = 35$. When assignments were evaluated in terms of costs, the greedy heuristic had difficulties on finding a feasible solution. For $m = 3$, the probability of succeeding was around 0.5. However, no feasible solution was found when $m = 5$. The pseudo-cost function proposed in Section 5.4.2, f_{λ^*} , shows an intermediate behaviour between choosing the costs and the requirements but not at the expense of a large error (see Figures 5.3 and 5.4). More precisely, we observe that the probability on succeeding when $m = 3$ stays around 0.8, and for $m = 5$ is around 0.6. Recall that the greedy heuristic with f_{λ^*} is asymptotically feasible if Assumptions 5.4.1 and 5.4.4 hold. Figures 5.1 and 5.2 suggest that Assumption 5.4.4 is necessary to ensure asymptotic feasibility of this greedy heuristic.

In Section 5.5.3 we will show that solving the GAP to optimality can be very time consuming. Therefore, the quality of the solution given by the greedy heuristic has been measured as the percentage by which the greedy heuristic value exceeds the optimal value of the LP-relaxation of the GAP. Figures 5.3 and 5.4 show the behaviour of the average error bound. This average error bound was calculated only for the problem instances where the greedy heuristic found a feasible solution. In opposite to the conclusions drawn about the feasibility of the greedy heuristic, we observe that the pseudo-cost functions a_{ij} and $(c_{ij} - 60)/a_{ij}$ provide feasible solutions with large error bound. When assignments are evaluated by the requirements, the average error bound is around 45% for $m = 3$ and 65% for $m = 5$. Those error bounds improve when using the pseudo-cost function $(c_{ij} - 60)/a_{ij}$, going to 17% when $m = 3$ and to 23% when $m = 5$. If assignments are evaluated by the costs, the average error bound is almost zero when $m = 3$. Recall that no feasible solution was found when $m = 5$. Finally, a good behaviour is illustrated for the pseudo-cost function $c_{ij} + \lambda_i^* a_{ij}$. The average error bound is below 1.5% when $m = 3$ and

decreases when n increases. For $m = 5$, it is below 6% and this average error bound is halved for $n \geq 55$. Figures 5.3 and 5.4 suggest that, under asymptotic feasibility, this greedy heuristic is asymptotically optimal.

Table 5.1 illustrates the robustness of the average error bound. Recall that the pseudo-cost function a_{ij} was able to find always a feasible solution. However, the probability of succeeding with the pseudo-cost function $c_{ij} + \lambda_i^* a_{ij}$ was around 0.8, and for $m = 5$ around 0.6. We would like to investigate whether the pseudo-cost function a_{ij} shows a better behaviour on the problem instances where the pseudo-cost function $c_{ij} + \lambda_i^* a_{ij}$ found a feasible solution. In Table 5.1, column er shows the average upper bound on the error with the pseudo-cost function a_{ij} , and column er^x is the average upper bound on the error of the problem instances where the pseudo-cost function $c_{ij} + \lambda_i^* a_{ij}$ succeeded in finding a feasible solution. We can conclude that the average upper bound on the error is robust, and therefore the results given above are not biased. Moreover, we can state that the solutions given by the pseudo-cost functions a_{ij} and $(c_{ij} - 60)/a_{ij}$ are far from optimality.

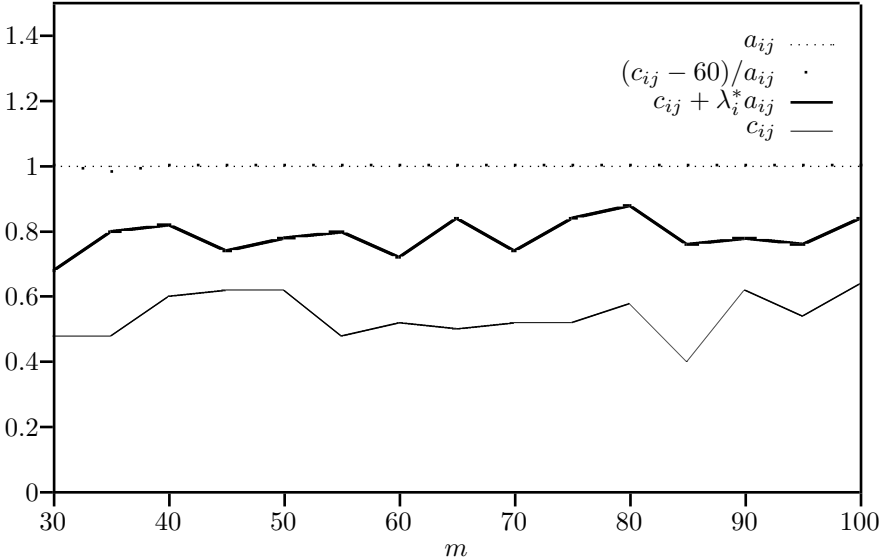


Figure 5.1: Average fraction of feasible solutions, $m = 3$

5.5.3 Improved heuristic

In this section we test the performance of the greedy heuristic with the pseudo-cost function f_{λ^*} and two local exchange procedures to improve the current solution. The guidelines of those two procedures were described in Section 2.3.2 for the CCAP. Particular choices were made in Section 4.4.2 for the GAP where the behaviour of this solution procedure (the greedy heuristic and the local exchange procedures) was illustrated for three different stochastic models.

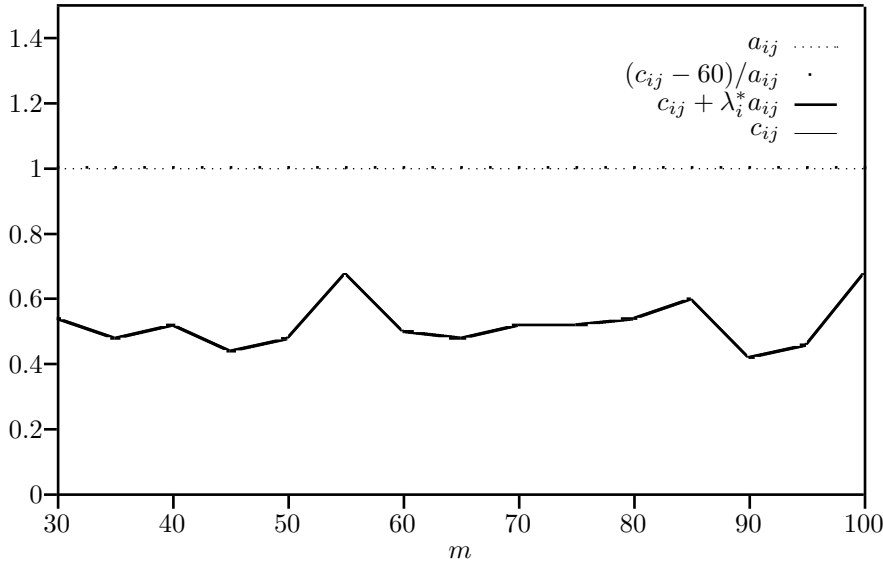


Figure 5.2: Average fraction of feasible solutions, $m = 5$

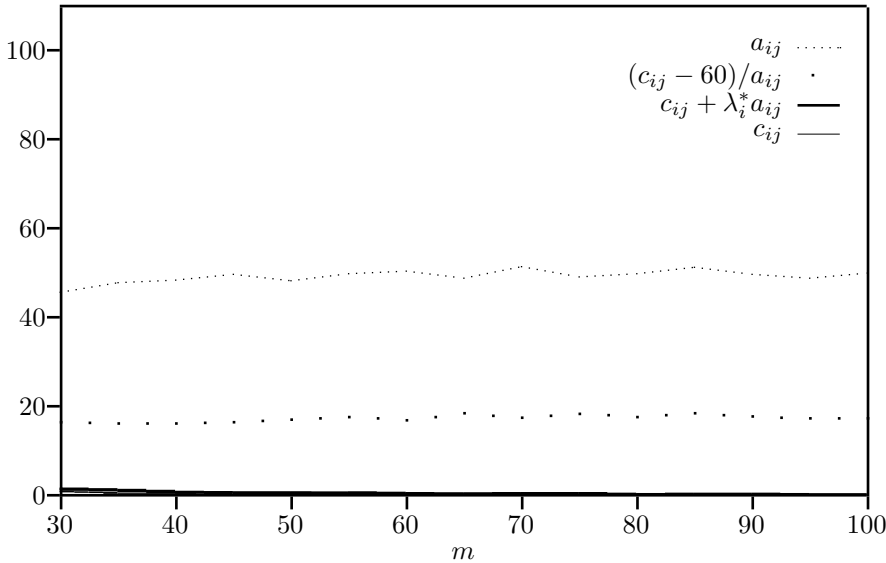


Figure 5.3: Average error bound, $m = 3$

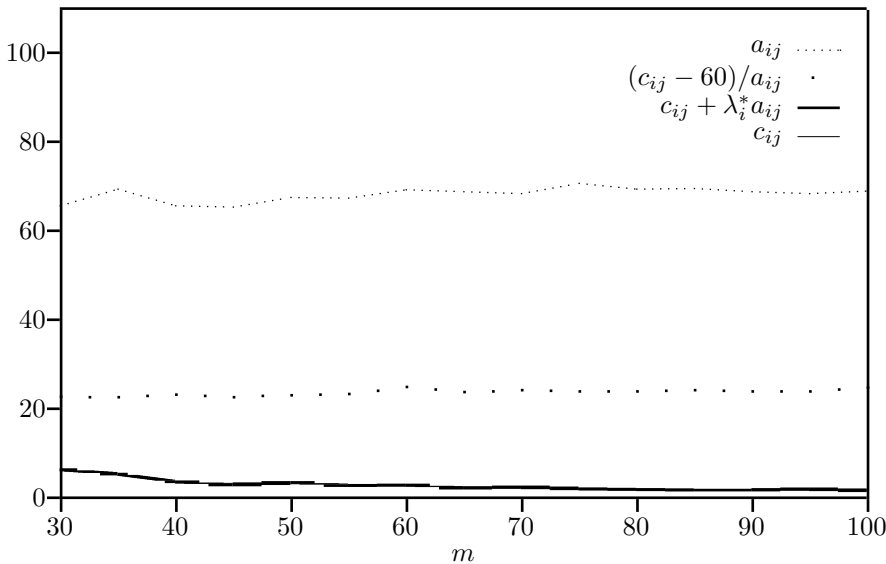


Figure 5.4: Average error bound, $m = 5$

n	$m = 3$		$m = 5$	
	er	er ^r	er	er ^r
30	45.69	44.98	65.60	67.97
35	47.74	47.54	69.35	67.77
40	48.32	48.07	65.68	67.30
45	49.71	49.95	65.34	67.41
50	48.28	47.89	67.54	69.82
55	49.81	50.58	67.34	67.77
60	50.34	50.62	69.30	69.54
65	48.75	47.83	68.79	68.62
70	51.43	50.30	68.39	67.71
75	49.12	49.15	70.76	71.02
80	49.84	49.78	69.36	70.23
85	51.24	49.94	69.54	68.36
90	49.68	49.32	68.87	68.65
95	48.82	47.82	68.39	69.80
100	49.90	49.27	69.03	69.83

Table 5.1: Average error bound for a_{ij} when $c_{ij} + \lambda_i^* a_{ij}$ is feasible

We refresh the choices we have made in Section 4.4.2 for the local exchange procedures. Recall that the first one tries to assign the tasks on which the greedy heuristic failed. The assignment of each of those tasks to agent i is measured by $r(i, j) = a_{ij}$ and the best agent, say i_j , is defined as the one minimizing $r(i, j)$. We try to assign the tasks to their most desirable agent in decreasing order of $r(i_j, j)$, either directly when agent i_j has sufficient capacity available, or by a feasible exchange, if one can be found. The second local exchange procedure tries to improve the objective value of the current solution. There, the possible exchanges of tasks (ℓ, p) are considered in decreasing order of

$$(f(i_\ell, \ell) + f(i_p, p)) - (f(i_\ell, p) + f(i_p, \ell))$$

where i_ℓ and i_p are the agents to which tasks ℓ and p are assigned in the current solution and $f(i, j) = c_{ij} + \lambda_i^* a_{ij}$ is the pseudo-cost function used by the greedy heuristic.

The parameters defining the problem instances were taken the same ones as in the previous section. We have chosen three different values of the number of agents $m = 5, 8$ and 10 , and the number of tasks was chosen to be equal to $5m, 6m, 7m$ and $8m$. For each problem size, 50 problem instances were generated.

Table 5.2 summarizes the results. Column I indicates the size of the problem, in the format $m.n$, where m represents the number of agents and n the numbers of tasks. Following this we have four groups of columns reporting information about the LP-relaxation (LPR), the greedy heuristic (G), the local exchange procedure to improve feasibility (F), and the local exchange procedure to improve the objective value (O). In this collection of problem instances for the GAP the LP-relaxation was always feasible. In block LPR, we have reported the average computation time used to solve the LP-relaxation. In block G, column st reports the status of the solution given by the greedy heuristic, more precisely, column st is equal to the number of problem instances for which the greedy heuristic could assign all the tasks, i.e., a feasible solution for the GAP was found. Column er is the average error bound (measured as the percentage by which the greedy heuristic value exceeds the LP-relaxation value). Obviously, this average was calculated only for the problem instances where a feasible solution was found. Column t is the average time employed by the greedy heuristic. Note that we need to solve the LP-relaxation to obtain the pseudo-cost function f_{λ^*} . We have reported the running time of the greedy heuristic without including the computation time required by the LP-relaxation. If the greedy heuristic could not assign all the tasks, we called the local exchange procedure to improve feasibility. Observe that this procedure was called the number of times that column $G-st$ lacks from 50. In block F, similar information as the one given for the greedy heuristic was reported for this procedure. Column st is the number of problem instances for which the procedure found a feasible solution for the GAP. Column er is the average error bound which was calculated only for the problem instances where a feasible solution was found. Column t is the average required time. For each problem instance where the greedy heuristic together with the local exchange procedure to improve feasibility found a feasible solution, we called the local exchange procedure to improve the objective value. In block O, we have reported the average error bound

and the average computation time. Finally, column *tt* indicates the average total time required by this solution procedure, i.e., the greedy heuristic together with the two local exchange procedures.

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
5.15	0.01	19	12.96	0.0002	26	26.29	0.0000	16.31	0.0002	0.05
5.20	0.01	24	8.05	0.0002	25	21.82	0.0000	11.32	0.0004	0.02
5.25	0.00	26	6.60	0.0002	24	15.01	0.0004	7.61	0.0004	0.02
5.30	0.01	21	6.11	0.0006	29	11.73	0.0000	6.51	0.0004	0.02
8.24	0.01	16	14.35	0.0004	31	27.16	0.0000	19.19	0.0002	0.03
8.32	0.01	21	11.47	0.0000	28	21.43	0.0000	12.55	0.0006	0.03
8.40	0.01	20	7.65	0.0002	29	15.43	0.0000	9.26	0.0008	0.03
8.48	0.02	21	6.44	0.0002	29	12.42	0.0000	7.09	0.0010	0.03
10.30	0.01	22	15.00	0.0006	20	27.10	0.0004	16.71	0.0005	0.03
10.40	0.02	17	11.67	0.0006	33	21.45	0.0000	14.59	0.0022	0.04
10.50	0.02	18	7.99	0.0006	32	16.24	0.0003	9.78	0.0008	0.04
10.60	0.03	18	6.59	0.0004	32	14.50	0.0000	8.30	0.0030	0.05

Table 5.2: Greedy heuristic + improvement phase; small ratios

We observe that the greedy heuristic together with the local exchange procedure to improve feasibility succeeded for almost all the problem instances to find a feasible solution. Note that as the ratio between the number of tasks and the number of agents (n/m) increases a feasible solution can be always ensured. Similarly, the error bound for the greedy heuristic and for the local exchange procedure to improve feasibility decrease when n/m increases. We observe that the greedy heuristic shows better error bounds than the local exchange procedure. The explanation is straightforward. While the greedy heuristic tries to combine the information about the costs and the requirements in the pseudo-cost function f_{λ^*} , the local exchange procedure to improve feasibility is myopic and only concentrates on the requirements. In the latter, the desirability of an assignment is evaluated by $r(i_j, j) = a_{i_j j}$. The error bound of the solution at hand after those two procedures can be calculated as the weighted average of the columns *G-er* and *F-er* with columns *G-st* and *F-st*. The quality of this solution is improved by the local exchange procedure for optimality. With respect to the computation times, reading data and solving the LP-relaxation are the main consumers.

The error bounds given in Table 5.2 seem to be quite high. To give an impression about their quality, we solved those problem instances to optimality with the MIP solver of CPLEX 6.5 [33]. Due to the hardness of the GAP, we allowed a maximal computation time of 30 minutes. Table 5.3 shows the results. Column *gap* represents the percentage of the optimal value of the GAP over the optimal value of the LP-relaxation. Column *tt* is the total time employed by the procedure. Finally, column $\#t^{\text{exc}}$ reports the number of times that optimality of the solution at hand could not be proved after 30 minutes.

The gap between the optimal solution value of the GAP and of its LP-relaxation

is considerable large. Note that it is above 11% when $n/m = 3$, and thus the error of the solution given by the solution procedure for the GAP drops below 9% for this ratio. Therefore, the error bounds given in Table 5.2 are not accurate for small size problem instances. The gap between the optimal solution for the GAP and for its LP-relaxation decreases when the ratio n/m increases illustrating then Theorem 5.4.3. From those results we also conclude that the MIP solver of CPLEX is able to solve relatively fast problem instances where the number of agents is small. However, the computation time drastically increases with the number of agents. For example, optimality of the solution at hand could not be proved for 90% of the problems instances where $m = 10$ and $n = 60$ after 30 minutes.

I	CPLEX		
	gap	tt	#t ^{exc}
5.15	11.22	0.44	0
5.20	7.24	0.72	0
5.25	4.45	1.57	0
5.30	3.43	3.46	0
8.24	11.12	48.45	0
8.32	6.80	123.42	0
8.40	5.04	400.34	3
8.48	3.63	715.04	12
10.30	11.02	380.53	4
10.40	7.29	1172.91	26
10.50	5.09	1485.65	36
10.60	3.73	1656.29	45

Table 5.3: Solving the GAP to optimality

Table 5.2 suggests that the performance of this solution procedure for the GAP improves when the ratio n/m increases. Similar information is reported in Table 5.4 when the ratio between the number of tasks and the number of agents was varied from 10 until 25 with increments of 5.

For this collection of problem instances of the GAP, a feasible solution was always found. The error bound was always below 4.25%, and below 1.75% when $n/m \geq 20$. Moreover, the computation time still stays moderate, in particular, it is always below 0.25 seconds.

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
5.50	0.01	24	3.56	0.0004	26	7.79	0.0000	3.62	0.0014	0.02
5.75	0.01	25	2.18	0.0004	25	5.27	0.0000	2.07	0.0054	0.03
5.100	0.02	23	1.55	0.0004	27	4.35	0.0000	1.28	0.0092	0.04
5.125	0.03	32	1.21	0.0002	18	3.28	0.0000	0.90	0.0148	0.06
8.80	0.03	21	4.01	0.0008	29	6.92	0.0000	3.80	0.0050	0.05
8.120	0.05	21	2.73	0.0006	29	5.05	0.0007	2.24	0.0144	0.08
8.160	0.07	23	1.85	0.0002	27	3.97	0.0007	1.55	0.0252	0.11
8.200	0.09	29	1.34	0.0024	21	2.88	0.0000	1.12	0.0438	0.15
10.100	0.05	24	4.29	0.0008	26	8.72	0.0000	4.18	0.0110	0.08
10.150	0.08	22	2.73	0.0010	28	5.80	0.0004	2.44	0.0234	0.12
10.200	0.13	22	2.02	0.0012	28	3.82	0.0004	1.66	0.0422	0.19
10.250	0.16	19	1.51	0.0014	31	3.20	0.0003	1.30	0.0712	0.25

Table 5.4: Greedy heuristic + improvement phase; big ratios

Part III

Supply Chain Optimization in a dynamic environment

Chapter 6

Multi-Period Single-Sourcing Problems

6.1 Introduction

The satisfaction of the demand for products of a set of customers involves several complex processes. In the past, this caused that both practitioners and researchers investigated those processes separately. As mentioned by Erengüç, Simpson and Vakharia [39], companies competing, for example with low prices, would sacrifice their flexibility in offering new products or satisfying new demands from their customers. The competition in the marketplace and the evolution of the hardware and software capabilities has offered both practitioners and researchers the possibility of considering the interaction between processes in the supply chain, thus attempting to integrate decisions concerning different functions. In this chapter we will propose a class of optimization models which integrate production, transportation and inventory decisions.

Most of the optimization models in the literature concerning the configuration of the logistics distribution network focus their attention on the location and size of production and the allocation of the demand of the customers, thus disregarding inventory control decisions. The classical model of Geoffrion and Graves [58] illustrates this issue. They analyze a three-level logistics distribution network composed of a set of production facilities, a set of possible locations for distribution centers and a set of customer zones. Upper bounds on the production levels are present, as well as lower and upper bounds on the throughput at each distribution center. Several products are requested by the customer zones and all of them have to be delivered by the same distribution center, or by the same production facility in the case that a direct shipment is required for each customer zone. They propose a single-period multicommodity production-distribution model to find the configuration of the distribution centers minimizing the production, transportation, and operating costs at the distribution centers. The latter costs are defined as the aggregation of a fixed cost due to the opening of the distribution center and a variable cost proportional to

its throughput.

A way of indirectly incorporating the inventory costs into the model is by expressing them as a function of the throughput at the warehouses. Fleischmann [49] uses a network design model to estimate the transportation and warehousing costs of a given layout of a logistics distribution network. The transportation costs are assumed to be a concave function of the quantity shipped, then modeling the economies of scale. The warehousing costs include inventory costs. To estimate these for a given warehouse, Fleischmann [49] argues that it suffices to express its stock level as a function of the throughput (demand) at the warehouse. He decomposes the stock level into working stock and safety stock. He proposes as an estimate for the working stock half of the replenishment but at least the daily throughput, i.e.,

$$\frac{1}{2} \max(r/N, \min(L, r/f))$$

where r is the throughput at the warehouse, N is the number of working days in the planning horizon, L is the full truck load, and f is the number of times that the warehouse is replenished. For the safety stock, he claims that \sqrt{r} is a common rough estimate.

There are some references in the literature where the inventory levels are explicitly modeled. Duran [37] studies a dynamic model for the planning of production, bottling, and distribution of beer, but focuses on the production process. Arntzen et al. [4] present a multi-echelon multi-period model to evaluate global supply chain configurations. They model the objective function as a convex combination of a costs term including production, inventory, taxes and net duty charges costs, and a term representing activity days.

Chan, Muriel and Simchi-Levi [29] highlight the importance of combining transportation and inventory decisions to achieve costs saving and improved service level in a supply chain. As does Fleischmann [49], they consider a network design model where the transportation costs are concave in the quantity shipped. In contrast to Fleischmann [49], their model is dynamic, but uncapacitated.

We propose a class of capacitated dynamic optimization models, called multi-period single-sourcing problems (hereafter MPSSP's), which can be used to answer strategic and tactical questions. In the first case, the MPSSP evaluates an estimate of the total costs of a given design of the logistics distribution network, including production, handling, inventory holding, and transportation costs. There, this evaluation is supposed to take place during a *typical* planning period in the future. Moreover, the MPSSP can also be used to answer tactical questions. The MPSSP is suitable for clustering customers with respect to the warehouses, and through this as the first step towards estimating operational costs in the network related to the daily delivery of the customers in tours. In this case, the planning horizon has particular start and end points.

The particular scenario we consider concerns a set of plants where a single product type is produced. The production in the plants is constrained due to their capacities. We do not allow for transportation between plants. A set of warehouses is used to facilitate the delivery of the demand to the customers. We assume that products are transported to the warehouses immediately, i.e., no storage is allowed at the

plants. When the products arrive at the warehouses they can be stored until a customer demand occurs. We do not allow for transportation between warehouses. The physical capacity of the warehouses, as well as their throughput, is limited. Customers are supplied by the warehouses. Customer service considerations lead to the so-called *single-sourcing* condition that each customer has to be delivered by exactly one warehouse (see Van Nunen and Benders [134], and Gelders, Pintelon and Van Wassenhove [56]).

Throughout this chapter and Chapters 7–9 we will assume that only plants face capacity constraints, and that production and storage take place at the same location. In Chapter 10 we will analyze the effect of including capacity constraints on the throughput at the warehouses, physical capacity constraints, and perishability constraints to model the shelf-life of the product. In Chapter 11 we will analyze a more general layout of the logistics distribution network where the production and the storage locations are decoupled, and thus a real interaction between the plants and the warehouses is allowed.

The reader should bear in mind that, unlike the term GAP, the term *multi-period single-sourcing problems* does not refer to a unique, well-defined problem, but to a class of problems.

The outline of this chapter is as follows. In Section 6.2 we will formulate the class of multi-period single-sourcing problems described above as mixed integer linear programming problems. In Section 6.3 we reformulate them as convex capacitated assignment problems. Finally, in Section 6.4 we derive some properties of the linear programming relaxation of their mixed integer linear formulation. Some of the results in this chapter can be found in Romeijn and Romero Morales [112, 113, 114].

6.2 The model

In this section we introduce a class of multi-period single-sourcing problems which will be analyzed throughout this chapter and Chapters 7–9. Recall that we are assuming that production and storage take place at the same location. Hereafter, we will refer to a facility as the combination of a plant and its corresponding warehouse.

Let n denote the number of customers, m the number of facilities, and T the planning horizon. The demand of customer j in period t is given by d_{jt} , while the production capacity at facility i in period t is equal to b_{it} . The unit production costs at facility i in period t are p_{it} , and the costs of assigning customer j to facility i in period t are a_{ijt} . Note that we make the (somewhat restrictive) assumption that the production costs are linear in the quantity produced. However, the assignment costs can be arbitrary functions of demand and distance. Finally, unit inventory holding costs at facility i in period t are equal to g_{it} . (All parameters are nonnegative by definition.)

As mentioned in the introduction, the MPSSP can be used for strategic and tactical purposes. In the first case, the costs are estimated during a typical planning period in the future. Therefore, a model is needed without a predefined beginning or end in the planning horizon. This can be achieved by assuming that the planning horizon represents an equilibrium situation, i.e., the planning period will repeat itself.

The demand pattern is then stationary with respect to the cycle length T . That is, $d_{j,T+1} = d_{j1}, d_{j,T+2} = d_{j2}, \dots$; in other words, the demand pattern is *cyclic* with period T . As a consequence, in equilibrium the inventory pattern at the facilities will (without loss of optimality) be cyclic as well. In the second case, i.e., when the MPSSP is used for tactical purposes, the planning horizon has a predefined beginning and end. In this case, we can assume without loss of generality that the starting inventory level is equal to zero while, without loss of optimality, the ending inventory will be equal to zero too (by the nonnegativity of the holding costs). Thus, the *cyclic* model will determine optimal starting (and ending) inventories while in the optimal solution of the *acyclic* one those are equal to zero. To be able to incorporate both the cyclic and the acyclic cases at the same time in the model, we introduce the set $\mathcal{C} \subseteq \{1, \dots, m\}$ of facilities at which the inventory pattern is restricted to be cyclic. It is clear that the only interesting and realistic cases are the two extremes $\mathcal{C} = \emptyset$ and $\mathcal{C} = \{1, \dots, m\}$. Therefore, we will pay particular attention to these two cases. Hereafter the indicator function $1_{\{Q\}}$ takes the value 1 if statement Q is true, and 0 otherwise.

Customer service considerations may necessitate that some or all customers are assigned to the same facility in each period. To incorporate this possibility into the model, we introduce the set $\mathcal{S} \subseteq \{1, \dots, n\}$ of customers (called static customers) that needs to be assigned to the same facility in all periods. We let $\mathcal{D} = \{1, \dots, n\} \setminus \mathcal{S}$ denote the remaining set of customers (called dynamic customers).

The problem can now be formulated as follows:

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m p_{it} y_{it} + \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt} + \sum_{t=1}^T \sum_{i=1}^m g_{it} I_{it}$$

subject to (P0)

$$\sum_{j=1}^n d_{jt} x_{ijt} + I_{it} = y_{it} + I_{i,t-1} \quad i = 1, \dots, m; t = 1, \dots, T \quad (6.1)$$

$$y_{it} \leq b_{it} \quad i = 1, \dots, m; t = 1, \dots, T \quad (6.2)$$

$$\sum_{i=1}^m x_{ijt} = 1 \quad j = 1, \dots, n; t = 1, \dots, T \quad (6.3)$$

$$x_{ijt} = x_{ij1} \quad i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \quad (6.4)$$

$$I_{i0} = I_{iT} 1_{\{i \in \mathcal{C}\}} \quad i = 1, \dots, m \quad (6.5)$$

$$y_{it} \geq 0 \quad i = 1, \dots, m; t = 1, \dots, T$$

$$x_{ijt} \in \{0, 1\} \quad i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \quad (6.6)$$

$$I_{it} \geq 0 \quad i = 1, \dots, m; t = 1, \dots, T$$

where y_{it} denotes the quantity produced at facility i in period t , x_{ijt} is equal to 1 if customer j is assigned to facility i in period t and 0 otherwise, and I_{it} denotes the inventory level at facility i at the end of period t . Constraints (6.1) model the

balance between the inflow, the storage and the outflow at facility i in period t . The maximal production capacity at facility i in period t is restricted by (6.2). Constraints (6.3) and (6.6) ensure that each customer is delivered by (assigned to) exactly one facility in each period. Moreover, constraints (6.4) ensure that each static customer is assigned to the same facility throughout the entire planning horizon. For each facility $i \in \mathcal{C}$, constraints (6.5) impose that the inventory levels at the beginning and at the end of the planning horizon are equal, and for each facility $i \notin \mathcal{C}$ that the inventory at the beginning of the planning horizon is equal to zero.

The MPSSP has been proposed to answer strategic and tactical questions in Supply Chain Management. Clearly, the most accurate answers are given by solving the MPSSP to optimality. However, finding the optimal value will be a formidable task due to the \mathcal{NP} -Hardness of the problem. This can easily be shown by considering the particular case where $T = 1$. Then the MPSSP reduces to the (single-period) Single-Sourcing Problem, which has been shown to be \mathcal{NP} -Hard by Fisher, Jaikumar and Van Wassenhove [48]). To even answer the question whether a given problem instance of the MPSSP has a feasible solution is an \mathcal{NP} -Complete problem. This follows again from the \mathcal{NP} -Completeness of the Single-Sourcing Problem, see Martello and Toth [88].

We can reduce this problem to the evaluation of the design of a two-level logistics distribution network by eliminating the production variables y_{it} . This yields the following reformulation of the problem:

$$\begin{aligned} & \text{minimize} \quad \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it} \\ & \text{subject to} \end{aligned} \tag{P}$$

$$\begin{aligned} \sum_{j=1}^n d_{jt} x_{ijt} + I_{it} & \leq b_{it} + I_{i,t-1} & i = 1, \dots, m; t = 1, \dots, T \\ \sum_{i=1}^m x_{ijt} & = 1 & j = 1, \dots, n; t = 1, \dots, T \\ x_{ijt} & = x_{ij1} & i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \\ I_{i0} & = I_{iT} 1_{\{i \in \mathcal{C}\}} & i = 1, \dots, m \\ x_{ijt} & \in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \\ I_{it} & \geq 0 & i = 1, \dots, m; t = 1, \dots, T \end{aligned}$$

where

$$c_{ijt} = a_{ijt} + p_{it} d_{jt}$$

and

$$h_{it} = p_{it} - p_{i,t+1} + g_{it}$$

(and $p_{i,T+1} \equiv p_{i1} 1_{\{i \in \mathcal{C}\}}$ for each $i = 1, \dots, m$).

Note that the inventory holding costs in (P) are not necessarily nonnegative (although they will be in the absence of speculative motives). This is in contrast with

an MPSSP that it is truly two-level in nature, where inventory holding costs would be nonnegative by definition. To be able to handle any MPSSP of the form (P), we will allow for virtually arbitrary inventory holding costs. The only restriction we will impose on these costs is that their aggregation over all periods is positive, i.e., $\sum_{t=1}^T h_{it} \geq 0$ for each $i \in \mathcal{C}$. As is shown in Proposition 6.2.1, this is a necessary and sufficient condition for the problem (P) to be well-defined, i.e., for its optimal value to be bounded from below. Clearly, this solution is satisfied if all inventory holding costs h_{it} are nonnegative. Observe that, for each $i \in \mathcal{C}$, $\sum_{t=1}^T h_{it} = \sum_{t=1}^T g_{it} \geq 0$, so that this condition is also satisfied if (P) is derived from the three-level problem (P₀).

For convenience, we have introduced the notation $[t] = (t + 1) \bmod T - 1$, i.e., $\alpha_{[t-1]} = \alpha_{t-1}$ for $t = 2, \dots, T$, and $\alpha_{[0]} = \alpha_T$.

Proposition 6.2.1 *Problem (P) is well-defined if and only if $\sum_{t=1}^T h_{it} \geq 0$ for each $i \in \mathcal{C}$.*

Proof: The feasible region of (P) is, in general, not bounded. This means that there may exist rays along which the feasible region is unbounded. We have to prove that the objective function is nondecreasing along those rays if and only if $\sum_{t=1}^T h_{it} \geq 0$ for each $i \in \mathcal{C}$.

Since the variables of (P) are nonnegative, so should the components of the rays. The assignment variables x_{ijt} are bounded as well as the inventory level variables for acyclic facilities, namely $I_{it} \in [0, \sum_{\tau=1}^t b_{i\tau}]$ for each $i \notin \mathcal{C}$ and $t = 1, \dots, T$. Thus, for each ray of the feasible region of (P), say $(r^1, r^2) \in \mathbb{R}_+^{mnT} \times \mathbb{R}_+^{mT}$, we have that $r_{ijt}^1 = 0$ for each $i = 1, \dots, m, j = 1, \dots, n, t = 1, \dots, T$ and $r_{it}^2 = 0$ for each $i \notin \mathcal{C}, t = 1, \dots, T$. We will show that $r_{it}^2 = r_{i1}^2$ for each $i \in \mathcal{C}, t = 2, \dots, T$. By the definition of a ray, for each vector (x, I) which belongs to the feasible region of (P), we have that

$$\sum_{j=1}^n d_{jt} x_{ijt} + I_{it} + \lambda r_{it}^2 \leq b_{it} + I_{i[t-1]} + \lambda r_{i[t-1]}^2$$

for each $\lambda \geq 0$, $i \in \mathcal{C}$, and $t = 1, \dots, T$, and thus

$$\sum_{j=1}^n d_{jt} x_{ijt} + I_{it} - b_{it} - I_{i[t-1]} \leq \lambda (r_{i[t-1]}^2 - r_{it}^2). \quad (6.7)$$

Since the vector (x, I) is feasible, the left hand-side of this inequality is nonpositive. Moreover, since the inequality should hold for all $\lambda \geq 0$, the ray should satisfy

$$r_{i[t-1]}^2 - r_{it}^2 \geq 0$$

for each $i \in \mathcal{C}$ and $t = 1, \dots, T$ which implies the desired result. From inequality (6.7) it is easy to see that each direction of this form, i.e. $(r^1, r^2) \in \mathbb{R}_+^{mnT} \times \mathbb{R}_+^{mT}$ so that $r_{ijt}^1 = 0$ for each $i = 1, \dots, m, j = 1, \dots, n, t = 1, \dots, T$, $r_{it}^2 = 0$ for each $i \notin \mathcal{C}, t = 1, \dots, T$ and $r_{it}^2 = r_{i1}^2$ for each $i \in \mathcal{C}, t = 2, \dots, T$, is a ray. In particular,

given $i' \in \mathcal{C}$, we can choose $r_{i't}^2 = 1$ for each $t = 1, \dots, T$ and $r_{it}^2 = 0$ for each $i \in \mathcal{C}$, $i \neq i'$ and $t = 1, \dots, T$. The directional derivative of the objective function at (x, I) along this ray is equal to $\sum_{t=1}^T h_{i't}$. The result follows by considering each $i' \in \mathcal{C}$. \square

Throughout this chapter we will study the two-level formulation (P) and will refer to c_{ijt} as the assignment costs and to h_{it} as the inventory holding costs.

6.3 Reformulation as a CCAP

6.3.1 Introduction

In this section we will show that (P) can be reformulated as a convex capacitated assignment problem. The optimization program (P) is posed in terms of assignment variables x_{ijt} , as well as inventory level variables I_{it} . (P) can be reformulated by replacing the inventory level variables by a nonlinear expression in the assignment variables. The advantage of this is that the problem can be viewed as a pure assignment problem. By decomposition techniques, we can easily see that the following is an equivalent formulation for (P):

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{i=1}^m H_i(x_{i..})$$

subject to

$$\begin{aligned} \sum_{i=1}^m x_{ijt} &= 1 & j = 1, \dots, n; t = 1, \dots, T \\ x_{ijt} &= x_{ij1} & i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \\ x_{ijt} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \end{aligned}$$

where the function $H_i(z)$, $z \in \mathbb{R}_+^{nT}$, is defined as the optimal value to the following linear programming problem:

$$\text{minimize } \sum_{t=1}^T h_{it} I_t$$

subject to

$$\begin{aligned} I_t - I_{t-1} &\leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt} & t = 1, \dots, T \\ I_0 &= I_T 1_{\{i \in \mathcal{C}\}} \\ I_t &\geq 0 & t = 1, \dots, T. \end{aligned}$$

Given a feasible set of assignments $x \in \mathbb{R}^{mnT}$, if for some $i = 1, \dots, m$ we have $H_i(x_{i..}) < +\infty$, then facility i is able to supply the demand required of this facility

by the vector of assignments x , i.e., $\sum_{j=1}^n d_{jt}x_{ijt}$ for each $t = 1, \dots, T$. Moreover, the value $H_i(x_{i..})$ is equal to the minimal inventory costs that facility i faces when supplying this demand.

The feasible region of this formulation is of the type of a CCAP. Its objective function is the sum of a linear term and the function $\sum_{i=1}^m H_i$. In the following sections we will prove that the problem is in fact a CCAP by showing that the objective function is convex.

6.3.2 The optimal inventory costs

The function H_i is unbounded for assignment vectors for which the required demand cannot be feasibly supplied due to the capacity constraints. We will refer to the domain of the function H_i as the set of vectors $z \in \mathbb{R}_+^{nT}$ where the function is well-defined, i.e., $H_i(z) < +\infty$. The next results show that the domain of the function H_i is defined by a set of linear constraints. We will study the cyclic and the acyclic case separately.

Lemma 6.3.1 *If $i \in \mathcal{C}$, the domain of the function H_i is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it} \right\}. \quad (6.8)$$

Proof: Consider some $z \in \mathbb{R}_+^{nT}$ in the domain of the function H_i . Then, there exists a vector $I' \in \mathbb{R}_+^T$ so that

$$I'_t - I'_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt}$$

for each $t = 1, \dots, T$ where $I_0 = I_T$. Aggregating those constraints over all the periods we obtain the desired inequality, and we can conclude that the domain of H_i is a subset of (6.8).

Now consider a vector $z \in \mathbb{R}_+^{nT}$ satisfying the condition

$$\sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it}. \quad (6.9)$$

If there exists a vector $y \in \mathbb{R}_+^T$ so that

$$y_t \leq b_{it} \quad t = 1, \dots, T \quad (6.10)$$

$$\sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} = \sum_{t=1}^T y_t, \quad (6.11)$$

then the vector $I = (I_t)$, where

$$I_t = \left(\sum_{\tau=1}^t y_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \right) - \min_{s=1, \dots, T} \left(\sum_{\tau=1}^s y_{i\tau} - \sum_{\tau=1}^s \sum_{j=1}^n d_{j\tau} z_{j\tau} \right)$$

for each $t = 1, \dots, T$ and $I_0 = I_T$, belongs to the feasible region of the LP problem defining the value $H_i(z)$. It is easy to check that I_t is nonnegative and

$$I_t - I_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt}$$

for each $t = 1, \dots, T$.

The existence of such a vector y can be trivially proved. By the inequality condition in (6.9), we know that there exists some $\tau = 1, \dots, T$ so that

$$\sum_{t=1}^{\tau-1} b_{it} < \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^{\tau} b_{it}.$$

Then we can define $y_t = b_{it}$ for $t = 1, \dots, \tau - 1$, $y_\tau = \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} - \sum_{t=1}^{\tau-1} b_{it}$ and $y_t = 0$ for $t = \tau + 1, \dots, T$. \square

In the next lemma we analyze the domain of the function H_i for an acyclic facility.

Lemma 6.3.2 *If $i \notin \mathcal{C}$, the domain of the function H_i is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=1}^t b_{i\tau}, t = 1, \dots, T \right\}. \quad (6.12)$$

Proof: Let $z \in \mathbb{R}_+^{nT}$ be a vector in the domain of the function H_i . Then, there exists a vector $I' \in \mathbb{R}_+^T$ so that

$$I'_t - I'_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt}$$

for each $t = 1, \dots, T$ where $I_0 = 0$. Aggregating the first t of those constraints, for each $t = 1, \dots, T$, we obtain the desired inequalities. Then, we can conclude that the domain of H_i is a subset of (6.12).

Now consider a vector $z \in \mathbb{R}_+^{nT}$ satisfying the conditions

$$\sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=1}^t b_{i\tau} \quad t = 1, \dots, T. \quad (6.13)$$

If we can prove that there exists a vector $y \in \mathbb{R}_+^T$ so that

$$y_t \leq b_{it} \quad t = 1, \dots, T \quad (6.14)$$

$$\sum_{\tau=1}^t y_\tau \geq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \quad t = 1, \dots, T, \quad (6.15)$$

then the vector $I = (I_t)$, where

$$I_t = \sum_{\tau=1}^t y_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau}$$

for each $t = 1, \dots, T$ and $I_0 = 0$, belongs to the feasible region of the LP problem defining the value $H_i(z)$. Observe that we can simply choose $y_t = b_{it}$ which can be shown to satisfy the inequalities in conditions (6.14) and (6.15) by using condition (6.13). \square

The following technical result prove some properties of the function H_i , that will be used to show that this function is convex and Lipschitz. For notational simplicity, let $h^+ = \max\{h, 0\}$.

Lemma 6.3.3 *For each z in the domain of the function H_i , the following statements hold:*

(i) *The value $H_i(z)$ can be rewritten as follows*

$$H_i(z) = \max_{\omega \in \Omega_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) \right\}$$

where

$$\begin{aligned} \Omega_i = \{ \omega \in \mathbb{R}_+^T : & h_{it} + \omega_t - \omega_{t+1} \geq 0, t = 1, \dots, T-1; \\ & h_{iT} + \omega_T - \omega_1 1_{\{i \in C\}} \geq 0 \}. \end{aligned} \quad (6.16)$$

(ii) *There exists $\omega^* \in \mathbb{R}_+^T$ so that*

$$H_i(z) = \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right),$$

and

$$\omega_t^* \leq \sum_{\tau=1}^{t-1} h_{i\tau}^+ + 1_{\{i \in C\}} \sum_{\tau=t+1}^T h_{i\tau}^+$$

for each $t = 1, \dots, T$.

Proof: By strong duality, we have that $H_i(z)$ can be written as follows

$$\begin{aligned} H_i(z) &= \\ &= \max_{\omega \in \mathbb{R}_+^T} \min_{I_t \geq 0, t=1, \dots, T; I_0 = I_T 1_{\{i \in C\}}} \left\{ \sum_{t=1}^T h_{it} I_t \right\} \end{aligned}$$

$$\begin{aligned}
& + \sum_{t=1}^T \omega_t \left(I_t - I_{t-1} - b_{it} + \sum_{j=1}^n d_{jt} z_{jt} \right) \Bigg\} \\
& = \max_{\omega \in \mathbb{R}_+^T} \min_{I_t \geq 0, t=1, \dots, T; I_0 = I_T \mathbf{1}_{\{i \in \mathcal{C}\}}} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) \right. \\
& \quad \left. + \sum_{t=1}^{T-1} (h_{it} + \omega_t - \omega_{t+1}) I_t - \omega_1 I_0 + (h_{iT} + \omega_T) I_T \right\} \\
& = \max_{\omega \in \Omega_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) \right\}
\end{aligned}$$

where Ω_i is defined by (6.16). This proves Claim (i).

To prove Claim (ii), we will consider the cyclic and the acyclic case separately. Let $i \in \mathcal{C}$. Let $\omega^* \in \Omega_i$ so that

$$H_i(z) = \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right).$$

We will first show that, without loss of optimality, there exists some t^* so that $\omega_{t^*}^* = 0$. Suppose that this does not hold and let e be the vector in \mathbb{R}^T whose components are all equal to one. It is easy to see that there exists $\varepsilon > 0$ so that $\omega^* - \varepsilon e \in \Omega_i$. Moreover, the objective function in $\omega^* - \varepsilon e$ is at least equal to the objective function in ω^* , i.e.,

$$\begin{aligned}
& \sum_{t=1}^T (\omega_t^* - \varepsilon) \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) = \\
& = \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) - \varepsilon \left(\sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} - \sum_{t=1}^T b_{it} \right) \\
& \geq \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right)
\end{aligned}$$

where the last inequality follows since $z \in \text{dom}(H_i)$. Thus, the desired result follows. Since $\omega_{t^*}^* = 0$, we have that

$$\omega_{t^*+1}^* \leq \omega_{t^*}^* + h_{it^*} = h_{it^*} = h_{it^*}^+$$

where the last inequality follows by the nonnegativity of $\omega_{t^*+1}^*$. In a recursive way, we can now easily prove that ω_t^* is not larger than

$$\begin{cases} \sum_{\tau=1}^{t-1} h_{i\tau}^+ + \sum_{\tau=t^*}^T h_{i\tau}^+ & \text{if } t = 1, \dots, t^* - 1 \\ \sum_{\tau=t^*}^T h_{i\tau}^+ & \text{if } t = t^* + 1, \dots, T \end{cases}$$

and Claim (ii) now follows easily for $i \in \mathcal{C}$.

It remains to prove Claim (ii) for $i \notin \mathcal{C}$. Let again $\omega^* \in \Omega_i$ so that

$$H_i(z) = \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right).$$

We would like to prove that, without loss of optimality, $\omega_1^* = 0$. Following a similar argument as for the cyclic case, there exists t^* so that $\omega_{t^*}^* = 0$. Suppose that t^* is the smallest index satisfying this condition. If $t^* = 1$ we are done, otherwise let e^{t^*-1} be the vector in \mathbb{R}_+^T so that $e_t^{t^*-1} = 1$ if $t = 1, \dots, t^* - 1$ and 0 otherwise. It is easy to see that there exists $\varepsilon > 0$ so that $\omega^* - \varepsilon e^{t^*-1} \in \Omega_i$. Moreover, the objective function in $\omega^* - \varepsilon e^{t^*-1}$ is at least equal to the objective function in ω^* , i.e.,

$$\begin{aligned} & \sum_{t=1}^T (\omega_t^* - \varepsilon e_t^{t^*-1}) \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) = \\ &= \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) - \varepsilon \left(\sum_{t=1}^{t^*-1} \sum_{j=1}^n d_{jt} z_{jt} - \sum_{t=1}^{t^*-1} b_{it} \right) \\ &\geq \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) \end{aligned}$$

where the last inequality follows since $z \in \text{dom}(H_i)$ for $i \notin \mathcal{C}$. Therefore, there exists $\hat{t} = 1, \dots, t^* - 1$ so that $\omega_{\hat{t}}^* = 0$. Suppose that \hat{t} is the smallest index satisfying this condition. If $\hat{t} = 1$, the desired result follows. Otherwise, we can repeat the same argument iteratively to show the result. Since $\omega_1^* = 0$, we have that

$$\omega_t^* \leq \sum_{\tau=1}^{t-1} h_{i\tau}^+$$

for each $t = 2, \dots, T$ and Claim (ii) follows for the acyclic case. \square

In the following result we derive two properties of the function H_i . First, we will show that this function is convex. Moreover, we will see that it is a Lipschitz function which formalizes the intuitive idea that the optimal inventory holding costs corresponding to two assignment solutions that nearly coincide should not differ by very much.

Proposition 6.3.4 *The function H_i is convex and Lipschitz.*

Proof: First we show that the function H_i is convex. From Lemmas 6.3.1 and 6.3.2, we have that the domain of the function H_i is the intersection of halfspaces in \mathbb{R}^{nT} , and thus a convex set. Now, let $\mu \in [0, 1]$, and z, z' be two vectors in the domain of the function H_i . Then, by Lemma 6.3.3(i), we have that

$$\begin{aligned}
 H_i(\mu z + (1 - \mu)z') &= \\
 &= \max_{\omega \in \Omega_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt}(\mu z_{jt} + (1 - \mu)z'_{jt}) - b_{it} \right) \right\} \\
 &= \max_{\omega \in \Omega_i} \left\{ \mu \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt}z_{jt} - b_{it} \right) \right. \\
 &\quad \left. + (1 - \mu) \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt}z'_{jt} - b_{it} \right) \right\} \\
 &\leq \mu \max_{\omega \in \Omega_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt}z_{jt} - b_{it} \right) \right\} \\
 &\quad + (1 - \mu) \max_{\omega \in \Omega_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt}z'_{jt} - b_{it} \right) \right\} \\
 &= \mu H_i(z) + (1 - \mu) H_i(z')
 \end{aligned}$$

which proves the convexity of the function H_i .

Recall that z and z' are in the domain of the function H_i . Without loss of generality we can assume that $H_i(z) \geq H_i(z')$. Moreover, consider $\omega^* \in \Omega_i$ for the vector z as defined in Lemma 6.3.3(ii). Then, we have that

$$\begin{aligned}
 |H_i(z) - H_i(z')| &= \\
 &= H_i(z) - H_i(z') \\
 &= \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt}z_{jt} - b_{it} \right) - \max_{\omega \in \Omega_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt}z'_{jt} - b_{it} \right) \right\} \\
 &\leq \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt}z_{jt} - b_{it} \right) - \sum_{t=1}^T \omega_t^* \left(\sum_{j=1}^n d_{jt}z'_{jt} - b_{it} \right) \\
 &= \sum_{t=1}^T \omega_t^* \sum_{j=1}^n d_{jt}(z_{jt} - z'_{jt}) \\
 &\leq \sum_{t=1}^T \omega_t^* \sum_{j=1}^n d_{jt}|z_{jt} - z'_{jt}|
 \end{aligned}$$

$$\begin{aligned}
&\leq L \sum_{t=1}^T \sum_{j=1}^n |z_{jt} - z'_{jt}| \\
&= L \|z - z'\|_1
\end{aligned}$$

where

$$L = \max_{j=1, \dots, n; t=1, \dots, T} d_{jt} \left(\sum_{\tau=1}^{t-1} h_{i\tau}^+ + 1_{\{i \in \mathcal{C}\}} \sum_{\tau=t+1}^T h_{i\tau}^+ \right). \quad (6.17)$$

Thus, the function H_i is Lipschitz. \square

6.3.3 An equivalent CCAP formulation

The reformulation of (P) given at the beginning of this section contains an objective function whose value may take on the value $+\infty$, i.e., the function $\sum_{i=1}^m H_i(x_{i..})$ is not necessarily well-defined for all the feasible vectors x . In the following theorem we prove that by adding the constraints defining the domain of the function H_i , for each $i = 1, \dots, m$, we obtain a CCAP formulation.

For notational simplicity, let $\text{dom}(H_i)$ denote the domain of the function H_i .

Theorem 6.3.5 *The reformulation of (P) given by*

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{i=1}^m H_i(x_{i..})$$

subject to (P')

$$\begin{aligned}
\sum_{i=1}^m x_{ijt} &= 1 & j = 1, \dots, n; t = 1, \dots, T \\
x_{ijt} &= x_{ij1} & i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \\
x_{ijt} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \\
x_{i..} &\in \text{dom}(H_i) & i = 1, \dots, m.
\end{aligned}$$

is a convex capacitated assignment problem.

Proof: From Lemmas 6.3.1 and 6.3.2, we have that the domain of the function H_i is given by linear constraints in the vectors $x_{i..} \in \mathbb{R}^{nT}$ for each $i = 1, \dots, m$, so the feasible region of (P') is a polyhedron. Moreover, the objective function is separable in the index i , and the costs associated with facility i are the aggregation of a linear term and a convex term in the vector $x_{i..}$, see Proposition 6.3.4. The desired result now follows. \square

As mentioned in Chapter 3, there are some multi-period single-sourcing problems which can be formulated as GAP's with nonlinear objective function. In particular,

if $\mathcal{C} = \{1, \dots, m\}$ and $\mathcal{S} = \{1, \dots, n\}$, and after eliminating the variables x_{ijt} for $i = 1, \dots, m$, $j \in \mathcal{S}$ and $t = 2, \dots, T$ and their corresponding assignment constraints, the reformulation of (P) given in the previous theorem is a GAP with convex costs. For general expressions of the sets \mathcal{C} and \mathcal{S} , we have proved that this reformulation is still a convex capacitated assignment problem.

Therefore, all the results we have developed in Chapter 2 for the class of convex capacitated assignment problems are also valid for the class of multi-period single-sourcing problems we have proposed in this chapter. In particular, in Chapter 7 we will analyze the generation of experimental data for this problem. In Chapter 8 we analyze the asymptotic behaviour of greedy heuristics for those problems. Finally, the Branch and Price procedure proposed for the CCAP is applied to some of these multi-period single-sourcing problems in Chapter 9.

6.4 The LP-relaxation

This section is devoted to the analysis of the LP-relaxation of (P) and resembles Section 3.5 where the LP-relaxation of the GAP was analyzed. Recall that the main result proved there, Proposition 3.5.2, was crucial when showing asymptotic feasibility and optimality of two greedy heuristics for the GAP. Similarly, the result of this section will be used in Chapter 8 to show asymptotic feasibility and optimality of a greedy heuristic for some variants of the MPSSP.

The linear programming relaxation (LPR) of (P) reads as follows:

$$\begin{aligned} & \text{minimize} \quad \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it} \\ & \text{subject to} \end{aligned} \tag{LPR}$$

$$\begin{aligned} \sum_{j=1}^n d_{jt} x_{ijt} + I_{it} & \leq b_{it} + I_{i,t-1} \\ & i = 1, \dots, m; t = 1, \dots, T \end{aligned} \tag{6.18}$$

$$\sum_{i=1}^m x_{ijt} = 1 \quad j = 1, \dots, n; t = 1, \dots, T \tag{6.19}$$

$$x_{ijt} = x_{ij1} \quad i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \tag{6.20}$$

$$I_{i0} = I_{iT} 1_{\{i \in \mathcal{C}\}} \quad i = 1, \dots, m$$

$$x_{ijt} \geq 0 \quad i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T$$

$$I_{it} \geq 0 \quad i = 1, \dots, m; t = 1, \dots, T.$$

Throughout this section we will assume that the feasible region of (LPR) is nonempty.

Benders and Van Nunen [14] give an upper bound on the number of infeasible tasks, i.e. the ones assigned to more than one agent, in the optimal solution of the LP-relaxation of the GAP. The following lemma derives a similar upper bound for

the class of MPSSP's analyzed in this chapter. Let $(x^{\text{LPR}}, I^{\text{LPR}})$ be a basic optimal solution for (LPR). For this solution, let B_S be the set of static customers such that $j \in B_S$ means that customer j is split (i.e., customer j is assigned to more than one facility, each satisfying part of its demand), and B_D be the set of (customer, period)-pairs such that $(j, t) \in B_D$ means that customer $j \in \mathcal{D}$ is split in period t .

Lemma 6.4.1 *Each basic optimal solution for (LPR) satisfies:*

$$|B_S| + |B_D| \leq mT.$$

Proof: Rewrite the problem (LPR) with equality constraints and nonnegativity variables only by introducing slack variables in (6.18), eliminating the variables x_{ijt} for $i = 1, \dots, m$, $j \in \mathcal{S}$ and $t = 2, \dots, T$, and variables I_{i0} for each $i = 1, \dots, m$. We then obtain a problem with, in addition to the assignment constraints, mT equality constraints. Now consider the optimal solution to (LPR). The number of variables having a nonzero value in this solution is no larger than the number of equality constraints in the reformulated problem. Since there is at least one nonzero assignment variable corresponding to each assignment constraint, and exactly one nonzero assignment variable corresponding to each assignment that is feasible with respect to the integrality constraints of (P), there can be no more than mT assignments that are split. \square

In the following lemma, which will be used in the proof of Proposition 6.4.3, we derive a relationship between the number of split assignments, the number of fractional assignment variables, the number of times a facility is used to full capacity in a period, and the number of strictly positive inventory variables. Let F_S be the set of fractional assignment variables in $(x^{\text{LPR}}, I^{\text{LPR}})$ associated with static customers (where each of these assignments is counted only for period 1, since the values of the assignment variables are equal for all periods) and F_D be the set of fractional assignment variables associated with dynamic customers, M be the set of (facility, period)-pairs such that $(i, t) \in M$ means that facility i is used to full capacity in period t , and I^+ be the set of strictly positive inventory variables in the vector I^{LPR} from period 1 until T . (Observe that I_{i0}^{LPR} is equal to I_{iT}^{LPR} or to 0.) These sets can be expressed as follows

$$\begin{aligned} F_S &= \{(i, j) : j \in \mathcal{S}, 0 < x_{ij1}^{\text{LPR}} < 1\} \\ F_D &= \{(i, j, t) : j \in \mathcal{D}, 0 < x_{ijt}^{\text{LPR}} < 1\} \\ M &= \{(i, t) : \sum_{j=1}^n d_{jt} x_{ijt}^{\text{LPR}} + I_{it}^{\text{LPR}} = b_{it} + I_{i,t-1}^{\text{LPR}}\} \\ I^+ &= \{(i, t) : t = 1, \dots, T, I_{it}^{\text{LPR}} > 0\}. \end{aligned}$$

Lemma 6.4.2 *If (LPR) is non-degenerate, then for each basic optimal solution of (LPR) we have that*

$$|F_S| + |F_D| + |I^+| = |M| + |B_S| + |B_D|.$$

Proof: Similarly as in the proof of Lemma 6.4.1, we can rewrite (LPR) with equality constraints and nonnegativity variables only by introducing slack variables in (6.18), say s_{it} , eliminating the variables x_{ijt} for $i = 1, \dots, m$, $j \in \mathcal{S}$ and $t = 2, \dots, T$, and variables I_{i0} for each $i = 1, \dots, m$. Let $(x^{\text{LPR}}, I^{\text{LPR}}, s^{\text{LPR}})$ be a basic optimal solution for the reformulation of (LPR). Then, the set M , defined above, is equal to

$$M = \{(i, t) : s_{it}^{\text{LPR}} = 0\}.$$

Under non-degeneracy, the number of nonzero variables at $(x^{\text{LPR}}, I^{\text{LPR}}, s^{\text{LPR}})$ is equal to $mT + |\mathcal{S}| + |\mathcal{D}| \cdot T$, the number of equality constraints in (LPR). The number of nonzero assignment variables is equal to $(|\mathcal{S}| - |B_{\mathcal{S}}|) + |F_{\mathcal{S}}| + (|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|) + |F_{\mathcal{D}}|$, where the first term corresponds to the variables $x_{ij1}^{\text{LPR}} = 1$ for $j \in \mathcal{S}$, the second one to the fractional assignment variables associated with static customers, analogously, the third term corresponds to the variables $x_{ijt}^{\text{LPR}} = 1$ for $j \in \mathcal{D}$, and the fourth one to the fractional assignment variables associated with dynamic customers. By definition $|I^+|$ is the number of nonzero inventory variables. With respect to the slack variables, we have $mT - |M|$ nonzero variables. Thus, by imposing that the number of nonzero variables at $(x^{\text{LPR}}, I^{\text{LPR}}, s^{\text{LPR}})$ is equal to $mT + |\mathcal{S}| + |\mathcal{D}| \cdot T$, we obtain

$$\begin{aligned} mT + |\mathcal{S}| + |\mathcal{D}| \cdot T &= \\ &= (|\mathcal{S}| - |B_{\mathcal{S}}|) + |F_{\mathcal{S}}| + (|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|) + |F_{\mathcal{D}}| + |I^+| + mT - |M|. \end{aligned}$$

The desired result now follows from the last equality. \square

After eliminating the variables x_{ijt} ($j \in \mathcal{S}$; $t = 2, \dots, T$) using equation (6.20), and removing equations (6.19) for $j \in \mathcal{S}$ and $t = 2, \dots, T$, the dual programming problem corresponding to (LPR) can be formulated as

$$\text{maximize } \sum_{j \in \mathcal{S}} v_j + \sum_{t=1}^T \sum_{j \in \mathcal{D}} v_{jt} - \sum_{t=1}^T \sum_{i=1}^m b_{it} \lambda_{it}$$

subject to

(D)

$$\begin{aligned} v_j &\leq \sum_{t=1}^T (c_{ijt} + \lambda_{it} d_{jt}) & i = 1, \dots, m; j \in \mathcal{S} \\ v_{jt} &\leq c_{ijt} + \lambda_{it} d_{jt} & i = 1, \dots, m; j \in \mathcal{D}; t = 1, \dots, T \\ \lambda_{i,t+1} - \lambda_{it} &\leq h_{it} & i = 1, \dots, m; t = 1, \dots, T-1 \\ \lambda_{i1} 1_{\{i \in \mathcal{C}\}} - \lambda_{iT} &\leq h_{iT} & i = 1, \dots, m \\ \lambda_{it} &\geq 0 & i = 1, \dots, m; t = 1, \dots, T \\ v_j &\text{ free} & j \in \mathcal{S} \\ v_{jt} &\text{ free} & j \in \mathcal{D}; t = 1, \dots, T. \end{aligned}$$

The next result characterizes the split assignments in the optimal solution for (LPR). This will be a crucial result when analyzing in Chapter 8 the asymptotic

feasibility and optimality of greedy heuristics for the class of multi-period single-sourcing problems presented here.

Proposition 6.4.3 *Suppose that (LPR) is non-degenerate. Let $(x^{\text{LPR}}, I^{\text{LPR}})$ be a basic optimal solution for (LPR) and let (λ^*, v^*) be the corresponding optimal solution for (D). Then,*

(i) *For each $j \in \mathcal{S} \setminus B_{\mathcal{S}}$, $x_{ijt}^{\text{LPR}} = 1$ for all $t = 1, \dots, T$ if and only if*

$$\sum_{t=1}^T (c_{ijt} + \lambda_{it}^* d_{jt}) = \min_{\ell=1, \dots, m} \sum_{t=1}^T (c_{\ell jt} + \lambda_{\ell t}^* d_{jt})$$

and

$$\sum_{t=1}^T (c_{ijt} + \lambda_{it}^* d_{jt}) < \min_{\ell=1, \dots, m; \ell \neq i} \sum_{t=1}^T (c_{\ell jt} + \lambda_{\ell t}^* d_{jt}).$$

(ii) *For each $j \in B_{\mathcal{S}}$, there exists i so that*

$$\sum_{t=1}^T (c_{ijt} + \lambda_{it}^* d_{jt}) = \min_{\ell=1, \dots, m; \ell \neq i} \sum_{t=1}^T (c_{\ell jt} + \lambda_{\ell t}^* d_{jt}).$$

(iii) *For each $(j, t) \notin B_{\mathcal{D}}$ so that $j \in \mathcal{D}$, $x_{ijt}^{\text{LPR}} = 1$ if and only if*

$$c_{ijt} + \lambda_{it}^* d_{jt} = \min_{\ell=1, \dots, m} (c_{\ell jt} + \lambda_{\ell t}^* d_{jt})$$

and

$$c_{ijt} + \lambda_{it}^* d_{jt} < \min_{\ell=1, \dots, m; \ell \neq i} (c_{\ell jt} + \lambda_{\ell t}^* d_{jt}).$$

(iv) *For each $(j, t) \in B_{\mathcal{D}}$, there exists i so that*

$$c_{ijt} + \lambda_{it}^* d_{jt} = \min_{\ell=1, \dots, m; \ell \neq i} (c_{\ell jt} + \lambda_{\ell t}^* d_{jt}).$$

Proof: Without loss of optimality, we can add to (D) the nonnegativity constraints on the vector v . By adding slack variables s_{ij} , s_{ijt} , and S_{it} to the constraints in (D), we can rewrite the dual problem with equality constraints and nonnegative variables.

Let $(\lambda^*, v^*, s^*, S^*)$ be the optimal solution for the new formulation. By the complementary slackness conditions, Claims (ii) and (iv) hold. To prove Claim (i), it suffices to show that for each $j \in \mathcal{S} \setminus B_{\mathcal{S}}$ there exists exactly one variable $s_{ij}^* = 0$. By complementary slackness conditions we know that at least there exists one of these variables. We have to show the uniqueness, and we do it by counting the variables at level zero in the vector $(\lambda^*, v^*, s^*, S^*)$. There are at least $mT - |M|$ variables λ_{it}^* , $|F_{\mathcal{S}}|$ variables s_{ij}^* corresponding to $j \in B_{\mathcal{S}}$, $|F_{\mathcal{D}}|$ variables s_{ijt}^* corresponding to $(j, t) \in B_{\mathcal{D}}$, $|\mathcal{S}| - |B_{\mathcal{S}}|$ variables s_{ij}^* corresponding to $j \in \mathcal{S} \setminus B_{\mathcal{S}}$, $|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|$

variables s_{ijt}^* corresponding to $(j, t) \notin B_{\mathcal{D}}$ and $j \in \mathcal{D}$, and $|I^+|$ variables S_{it}^* equal to zero. In total, we have at least

$$\begin{aligned} & mT - |M| + |F_{\mathcal{S}}| + (|\mathcal{S}| - |B_{\mathcal{S}}|) + |F_{\mathcal{D}}| + (|\mathcal{D}| \cdot T - |B_{\mathcal{D}}|) + |I^+| = \\ = & mT + |\mathcal{S}| + |\mathcal{D}| \cdot T \end{aligned}$$

zeroes in the optimal dual solution, where the last equality follows from Lemma 6.4.2. Therefore, these are exactly all the variables at level zero in vector $(\lambda^*, v^*, s^*, S^*)$. Then, for each $j \in \mathcal{S} \setminus B_{\mathcal{S}}$ there exists exactly one variable $s_{ij}^* = 0$, and Claim (i) follows. Claim (iii) now follows in a similar way. \square

Chapter 7

Feasibility analysis of the MPSSP

7.1 Introduction

A class of multi-period single-sourcing problems has been introduced in Chapter 6 which is suitable for both strategic and tactical purposes. Our goal is to solve those optimization models efficiently by means of both exact and heuristics solution procedures. The behaviour of those solution procedures will be thoroughly illustrated by testing them on a collection of randomly generated problem instances. In Chapter 4 we already discussed the relevance of the analysis of the test problem instances. There, we illustrated how different generators of random problem instances for the GAP yielded different conclusions on the performance of the same greedy heuristic for the GAP. As for the GAP, we devote this chapter to the analysis of the tightness of the problem instances obtained by a random generator for the MPSSP.

Two types of formulations have been given in Chapter 6 for the MPSSP, namely the standard one where the allocation of the customers and the location and sizes of the inventory levels have to be decided and a nonlinear one where the inventory holding costs are expressed as a nonlinear function of the vector of assignments. Throughout this chapter we will work with the second one, say (P') , given by Theorem 6.3.5. In Chapter 6, we already proved that (P') is a convex capacitated assignment problem. In the following we will go more into detail on this proof which will be helpful when making use of the feasibility conditions derived in Chapter 2 for the CCAP.

Recall that in the convex capacitated assignment problem we have a set of tasks and a set of agents which can process those tasks. Each task has to be processed by exactly one agent, and capacity constraints are faced by the agents when processing the tasks. In the MPSSP, the facilities can be seen as the agents, and the assignments associated with the customers as the tasks. There, we have two types of agents and two types of tasks. The agents can be cyclic or acyclic facilities depending on whether they show a cyclic inventory pattern or not. Each cyclic facility faces just one capacity

	static	dynamic
cyclic	$\sum_{t=1}^T d_{jt}$	d_{jt}
acyclic	$(d_{j1}, \dots, \sum_{t=1}^{\tau} d_{jt}, \dots, \sum_{t=1}^T d_{jt})^{\top}$	$(0, \dots, 0, d_{jt}, \dots, d_{jt})^{\top}$

Figure 7.1: Requirements in the CCAP formulation

constraint while each acyclic one faces T capacity constraints (recall that T is the planning horizon). In the first case, the only capacity constraint restricts the total flow through all periods while in the second case the cumulative flow through the first t periods is restricted, for each $t = 1, \dots, T$.

The assignments associated with the customers can be seen as the tasks and they can also be divided into two groups, namely, the ones associated with static customers and the ones with dynamic customers. Recall that a static customer must be assigned to the same facility for all periods. Therefore, each static customer induces exactly one task, while a dynamic one induces T tasks, each one representing the assignment of this customer in some period t , for each $t = 1, \dots, T$. The tasks require capacity available at the agents. Those requirements depend on the type of agent and the type of task. Let j be a customer and t a period. If the customer j is static, then we have exactly one task associated with this assignment. If the facility is cyclic, then there is exactly one capacity constraint restricting the total flow through that facility. Therefore, the requirement of task j is equal to $\sum_{t=1}^T d_{jt}$ where we recall that d_{jt} is the demand of customer j in period t . When the facility is acyclic, the cumulative flow over the first t periods is constrained for each $t = 1, \dots, T$, and thus the requirement of task j for the t -th capacity constraint is equal to $\sum_{\tau=1}^t d_{j\tau}$. While a static customer induces exactly one task in the convex capacitated assignment formulation, a dynamic one brings T new tasks, each one representing the facility supplying its demand in some period t , for each $t = 1, \dots, T$. We will denote those tasks by the (customer,period)-pairs (j, t) . Again their requirements depend on the type of facility. If the facility is cyclic, task (j, t) requires d_{jt} , while if it is acyclic it requires 0 in the first $t - 1$ constraints and d_{jt} for the τ -th capacity constraint, for all $\tau = t, \dots, T$. The requirements are illustrated by Figure 7.1 where we can find their expression depending on the type of facility (cyclic and acyclic) and the type of task (associated with a static or with a dynamic customer).

The outline of this chapter is as follows. In Section 7.2 we describe a stochastic model for the MPSSP. In the following two sections we probabilistically analyze the feasibility of the problem instances generated by this stochastic model. In Section 7.3 we find an explicit condition to ensure asymptotic feasibility in the probabilistic sense for the cyclic case. In Section 7.4 we are able to find explicit conditions for different subclasses of acyclic MPSSP's. Finally, Section 7.5 presents some numerical results to illustrate those feasibility conditions. Some of the results in this chapter can be

found in Romeijn and Romero Morales [112, 113, 114].

7.2 Stochastic model for the MPSSP

In this section we propose a stochastic model for the MPSSP. In the rest of the chapter we will analyze the tightness of the problem instances generated by this stochastic model.

Consider the following probabilistic model for the parameters defining the feasible region of a multi-period single-sourcing problem. Recall that \mathcal{S} is the set of static customers. For each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \overline{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

Furthermore, let b_{it} depend linearly on n , i.e., $b_{it} = \beta_{it}n$, for positive constants β_{it} . Observe that the type of facility (cyclic or acyclic) does not influence the generation of parameters for the stochastic model.

When the MPSSP is formulated as a convex capacitated assignment problem, this stochastic model can also be formulated as one of the stochastic models for the CCAP proposed in Section 2.2, for the particular case of the MPSSP. We will illustrate the cyclic case, and similarly can be done for other cases. In the general mixed case, when both static and dynamic customers are present, subsets of tasks are generated of different size as for the stochastic model for the CCAP given in Section 2.2.4. When $\gamma_j = 0$ we just generate a task, and T tasks when $\gamma_j = 1$. The former happens with probability π and the later with probability $1 - \pi$. Now we will investigate the expression of the vector of requirements in both cases. Let \mathbf{A}_{j01} be the vector of requirements if $\gamma_j = 0$, and \mathbf{A}_{j1t} be if $\gamma_j = 1$ and $t = 1, \dots, T$. Then, we have that $\mathbf{A}_{j01} = \left(\sum_{t=1}^T \mathbf{D}_{jt}, \dots, \sum_{t=1}^T \mathbf{D}_{jt} \right)^\top$ and $\mathbf{A}_{j1t} = (\mathbf{D}_{jt}, \dots, \mathbf{D}_{jt})^\top$ for each $t = 1, \dots, T$. From the assumptions on the stochastic model for the MPSSP, the size of the subsets of tasks ($j = 1, \dots, n$) is Bernoulli-distributed with parameter π which is a particular case of a multinomial distribution. Moreover, the vectors $(\mathbf{A}_{j01}, (\mathbf{A}_{j1t})_{t=1, \dots, T})$ ($j = 1, \dots, n$) are i.i.d.. Therefore, this model satisfies the conditions required for the stochastic model for the CCAP given in Section 2.2.4.

The feasibility analysis performed in Chapter 2 to the CCAP also applies to (P'). Recall that \mathcal{C} is the set of facilities where the inventory pattern is cyclic. There we defined the *excess capacity*, Δ , (see Theorem 2.2.3) and proved that, as $n \rightarrow \infty$, the CCAP is feasible with probability one if $\Delta > 0$, and infeasible with probability one if $\Delta < 0$. For (P'), the excess capacity reads

$$\begin{aligned} \Delta = & \min_{\lambda \in \mathcal{S}} \left(\sum_{i=1}^m \lambda_i^\top \bar{\beta}_i - \pi \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_1^{\mathcal{S}} \right) \right. \\ & \left. - (1 - \pi) \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \lambda_i^\top \mathbf{A}_{(1,t)}^{\mathcal{D}} \right) \right) \end{aligned} \quad (7.1)$$

where $\bar{\beta}_i = \sum_{t=1}^T \beta_{it}$ for each $i \in \mathcal{C}$ and $\bar{\beta}_i = (\bar{\beta}_{it}) = (\sum_{\tau=1}^t \beta_{i\tau})$ for each $i \notin \mathcal{C}$, \mathbf{A}_ℓ^S is the vector of requirements of task ℓ when $\ell = j \in \mathcal{S}$, \mathbf{A}_ℓ^D is the vector of requirements of task ℓ when $\ell = (j, t)$ and $j \in \mathcal{D}$, S is the unit simplex in $\mathbb{R}^{k_1} \times \dots \times \mathbb{R}^{k_m}$, and k_i is equal to 1 if the facility i is cyclic and to T if it is acyclic. We already pointed out in that chapter that the condition $\Delta > 0$ is implicit and implies finding the optimal solution of a nonlinear minimization problem. In the following, we will find explicit feasibility conditions for different classes of multi-period single-sourcing problems.

As mentioned in Chapter 6, the only realistic cases for the set \mathcal{C} are when all facilities are of the same type, i.e., all facilities are cyclic ($\mathcal{C} = \{1, \dots, m\}$) or all are acyclic ($\mathcal{C} = \emptyset$). In the following, we will analyze separately the two cases and we will skip the mixed one.

7.3 Explicit feasibility conditions: the cyclic case

In this section, we analyze strategic multi-period single-sourcing problems, i.e., the MPSSP when $\mathcal{C} = \{1, \dots, m\}$. The following theorem shows that to ensure asymptotic feasibility in the probabilistic sense we then simply need to impose that the total production capacity over all periods that is available per customer is larger than the total expected demand per customer over all periods.

Theorem 7.3.1 *If $\mathcal{C} = \{1, \dots, m\}$, (P') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) < \sum_{t=1}^T \sum_{i=1}^m \beta_{it}, \quad (7.2)$$

and infeasible with probability one if this inequality is reversed.

Proof: We will first show that condition (7.2) is equivalent to the excess capacity Δ being strictly positive. In this case, the excess capacity reads

$$\begin{aligned} \Delta &= \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i \left(\sum_{t=1}^T \beta_{it} \right) - \pi \mathcal{E} \left(\min_{i=1, \dots, m} \left(\lambda_i \sum_{t=1}^T \mathbf{D}_{1t} \right) \right) \right. \\ &\quad \left. - (1 - \pi) \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} (\lambda_i \mathbf{D}_{1t}) \right) \right) \\ &= \min_{\lambda \in S} \left(\sum_{i=1}^m \lambda_i \left(\sum_{t=1}^T \beta_{it} \right) - \left(\min_{i=1, \dots, m} \lambda_i \right) \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \right) \end{aligned}$$

where S is the unit simplex in \mathbb{R}^m .

Note first that the vector $\lambda_i = \frac{1}{m}$, for all $i = 1, \dots, m$, belongs to the set S . Thus, $\Delta > 0$ implies the condition in the theorem, which is therefore necessary. To prove sufficiency, we will show that condition (7.2) implies that the expression to be minimized is strictly positive for all $\lambda \in S$ (since S is compact). First consider vectors $\lambda \in S$ for which at least one element is equal to zero. Then the relevant

expression reduces to $\sum_{i=1}^m \lambda_i \left(\sum_{t=1}^T \beta_{it} \right)$, which is clearly positive, since all β_{it} 's are positive, and at least one λ_i is positive. So it remains to verify that the expression is positive for all vectors $\lambda \in S$ for which $\lambda_{\min} \equiv \min_{i=1, \dots, m} \lambda_i > 0$. For those λ 's, we have

$$\begin{aligned}
 & \sum_{i=1}^m \lambda_i \left(\sum_{t=1}^T \beta_{it} \right) - \left(\min_{i=1, \dots, m} \lambda_i \right) \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) = \\
 &= \sum_{i=1}^m \lambda_i \left(\sum_{t=1}^T \beta_{it} \right) - \lambda_{\min} \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \\
 &\geq \sum_{i=1}^m \lambda_{\min} \left(\sum_{t=1}^T \beta_{it} \right) - \lambda_{\min} \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \\
 &= \lambda_{\min} \left(\sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \right) \\
 &> 0
 \end{aligned}$$

by the assumption in the theorem, which shows the sufficiency of the condition.

Finally, if the inequality in (7.2) is reversed, it is easy to see that $\Delta < 0$ by considering $\lambda_i = \frac{1}{m}$, for all $i = 1, \dots, m$. \square

For the static case, this condition directly follows from the analysis developed for the GAP by Romeijn and Piersma [110]. Observe that when all customers are static, the feasible region of (P') can be seen as the one of a GAP with m agents and n tasks. The capacity of each agent is equal to $\sum_{t=1}^T b_{it}$ and the requirement of each task is equal to $\sum_{t=1}^T d_{jt}$. Observe that the requirements are agent-independent. For that particular case, Romeijn and Piersma [110] derived the explicit feasibility condition saying that the expected requirement of a task must be below the total relative capacity. This is precisely the condition we have obtained in Theorem 7.3.1.

7.4 Explicit feasibility conditions: the acyclic case

7.4.1 Introduction

In the previous section we have found an explicit tight condition to ensure feasibility of the problem instances in the probabilistic sense when all facilities have a cyclic inventory pattern. This section is devoted to the analysis of the feasibility of the problem instances when all facilities are acyclic. Unfortunately, we have not been able to derive explicit conditions for all cases. The complication comes from the static customers which have to be assigned to a single facility throughout the planning horizon, yielding a truly dynamic model, whereas the cyclic variant can be reformulated as an essentially static model. Due to the difficulty of the acyclic problem in the presence of static customers we are not able to find explicit feasibility conditions for the class of all problem instances of (P') with static customers. However, we are able

to find such conditions for the case of only dynamic customers, as well as for two large subclasses of problems of the static case.

For the acyclic case, the excess capacity reads as follows

$$\begin{aligned} \Delta = \min_{\lambda \in S} & \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \pi \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \right. \\ & \left. - (1 - \pi) \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \mathbf{D}_{1t} \right) \right) \end{aligned}$$

where S is the unit simplex in \mathbb{R}^{mT} .

The following result will be useful when analyzing mixed cases where dynamic and static customers may be present. Intuition suggests that the excess capacity for the static case should be at most equal to the excess capacity for the dynamic one. Moreover, the excess capacity for the mixed case with dynamic and static customers should be between the two extreme cases. This is formalized in the following result. There, we will make use of the notation Δ^S and Δ^D indicating the excess capacity for the static and the dynamic case, respectively. We will use Δ for the mixed case.

Lemma 7.4.1 *It holds*

$$\Delta^S \leq \Delta \leq \Delta^D.$$

Proof: Observe that

$$\Delta = \min_{\lambda \in S} (\pi f_S(\lambda) + (1 - \pi) f_D(\lambda))$$

where

$$\begin{aligned} f_S(\lambda) &= \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \\ f_D(\lambda) &= \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \mathbf{D}_{1t} \right). \end{aligned}$$

Moreover, $\Delta^S = \min_{\lambda \in S} f_S(\lambda)$ and $\Delta^D = \min_{\lambda \in S} f_D(\lambda)$. Thus, it suffices to show that $f_S(\lambda) \leq f_D(\lambda)$ for all $\lambda \in S$.

Now, for each $\lambda \in S$, we have that

$$\begin{aligned} f_S(\lambda) &= \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \\ &= \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{t=1}^T \mathbf{D}_{1t} \sum_{\tau=t}^T \lambda_{i\tau} \right) \\ &\leq \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \mathcal{E} \left(\sum_{t=1}^T \min_{i=1, \dots, m} \mathbf{D}_{1t} \sum_{\tau=t}^T \lambda_{i\tau} \right) \\ &= f_D(\lambda), \end{aligned}$$

and the desired inequality follows. \square

7.4.2 Only dynamic assignments

In this section we analyze the acyclic MPSSP where all customers are dynamic. We will show in the following theorem that to ensure asymptotic feasibility in the probabilistic sense we need to impose that the total production capacity over the t first periods that is available per customer is larger than the total expected demand per customer over the t first periods, for all $t = 1, \dots, T$.

Theorem 7.4.2 *If $\mathcal{C} = \emptyset$ and $\mathcal{S} = \emptyset$, (P') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) < \sum_{\tau=1}^t \sum_{i=1}^m \beta_{i\tau} \quad \text{for } t = 1, \dots, T \quad (7.3)$$

and infeasible with probability one if at least one of those inequalities is reversed.

Proof: We will first show that condition (7.3) is equivalent to the excess capacity Δ being strictly positive. In this case, the excess capacity reads

$$\begin{aligned} \Delta &= \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \mathbf{D}_{1t} \right) \right) \\ &= \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \right) \right) \\ &= \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \beta_{it} \left(\sum_{\tau=t}^T \lambda_{i\tau} \right) - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \right) \right) \end{aligned}$$

where S is the unit simplex in \mathbb{R}^{mT} .

First observe that the vectors $\lambda^{(\bar{\tau})}$ defined as

$$\lambda_{it}^{(\bar{\tau})} = \begin{cases} \frac{1}{m} & i = 1, \dots, m; t = \bar{\tau} \\ 0 & \text{otherwise,} \end{cases}$$

for each $\bar{\tau} = 1, \dots, T$, belong to S . Then $\Delta > 0$ implies the condition given by the theorem, and it follows that this condition is necessary.

By setting $\mu_{it} = \sum_{\tau=t}^T \lambda_{i\tau}$ for each $i = 1, \dots, m$ and $t = 1, \dots, T$, we can rewrite Δ as

$$\Delta = \min_{\mu \in S'} \left(\sum_{t=1}^T \sum_{i=1}^m \mu_{it} \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \min_{i=1, \dots, m} \mu_{it} \right)$$

where

$$S' = \left\{ \mu \in \mathbb{R}_+^{mT} : \sum_{i=1}^m \mu_{i1} = 1; \mu_{it} \geq \mu_{i,t+1}, t = 1, \dots, T-1 \right\}.$$

Since S' is compact, the sufficiency of condition (7.3) follows if that condition implies that

$$\sum_{t=1}^T \sum_{i=1}^m \mu_{it} \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \min_{i=1, \dots, m} \mu_{it} > 0 \quad (7.4)$$

for all $\mu \in S'$. Now let $S'_0 = \{\mu \in S' : \min_{i=1, \dots, m} \mu_{i1} = 0\}$ and $S'_1 = \{\mu \in S' : \min_{i=1, \dots, m} \mu_{i1} > 0\}$, so that $S' = S'_0 \cup S'_1$. In order to prove that (7.4) holds for all $\mu \in S'$, we will consider the cases $\mu \in S'_0$ and $\mu \in S'_1$ separately.

First, let $\mu \in S'_0$. Then, since

$$\min_{i=1, \dots, m} \mu_{it} \geq \min_{i=1, \dots, m} \mu_{i, t+1}$$

for all $t = 1, \dots, T-1$, we know that $\min_{i=1, \dots, m} \mu_{it} = 0$ for all $t = 1, \dots, T$. Thus,

$$\sum_{t=1}^T \sum_{i=1}^m \mu_{it} \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \min_{i=1, \dots, m} \mu_{it} = \sum_{t=1}^T \sum_{i=1}^m \mu_{it} \beta_{it} > 0.$$

Next, consider $\mu \in S'_1$. We have that

$$\begin{aligned} & \sum_{t=1}^T \sum_{i=1}^m \mu_{it} \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \min_{i=1, \dots, m} \mu_{it} \geq \\ & \geq \sum_{t=1}^T \sum_{i=1}^m \left(\min_{\ell=1, \dots, m} \mu_{\ell t} \right) \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \min_{i=1, \dots, m} \mu_{it} \end{aligned}$$

for all $\mu \in S'_1$. Now

$$\sum_{t=1}^T \sum_{i=1}^m \left(\min_{\ell=1, \dots, m} \mu_{\ell t} \right) \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \min_{i=1, \dots, m} \mu_{it} > 0$$

if and only if

$$\sum_{t=1}^T \sum_{i=1}^m \left(\frac{\min_{\ell=1, \dots, m} \mu_{\ell t}}{\min_{\ell=1, \dots, m} \mu_{\ell 1}} \right) \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \left(\frac{\min_{i=1, \dots, m} \mu_{it}}{\min_{\ell=1, \dots, m} \mu_{\ell 1}} \right) > 0.$$

This is true for all $\mu \in S'_1$ if

$$\min_{\delta \in S''} \left(\sum_{t=1}^T \left(\sum_{i=1}^m \beta_{it} \right) \delta_t - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \delta_t \right) > 0 \quad (7.5)$$

where

$$S'' = \{\delta \in \mathbb{R}_+^T : \delta_1 = 1; \delta_t \geq \delta_{t+1}, t = 1, \dots, T-1\}.$$

Since the minimization problem (7.5) is a linear programming problem, we can restrict the feasible region to the set of extreme points of S'' . These are given by

$$\delta_t^{(\tau)} = \begin{cases} 1 & t = 1, \dots, \tau \\ 0 & t = \tau + 1, \dots, T \end{cases}$$

for all $\tau = 1, \dots, T$ (see Carrizosa et al. [22]). Thus, condition (7.4) holds for all $\mu \in S'$ and then the sufficiency of condition (7.3).

Finally, suppose that, for some $\bar{\tau} = 1, \dots, T$, the inequality in (7.3) is reversed. Then it is easy to see that $\Delta < 0$ by choosing the corresponding vector $\lambda^{(\bar{\tau})}$ defined above. \square

Comparing condition (7.3) with the corresponding condition (7.2) in the cyclic case, we see that they are very similar. The difference is that, in the acyclic case, we need to impose a condition on the cumulative aggregate capacity for each planning horizon $t = 1, \dots, T$, instead of just for the full planning horizon T , as is sufficient in the cyclic case. This makes sense, since we have lost the option to essentially be able to produce in “later” (modulo T) periods for usage in “earlier” (modulo T) periods.

7.4.3 Identical facilities

In this section we analyze the special case of (P') with identical facilities, i.e., $\beta_{it} = \beta_t$ for each $i = 1, \dots, m$ and $t = 1, \dots, T$. In Theorem 7.4.4 we will show that the same condition as for the dynamic case holds.

The following lemma will be used in the proof of Theorem 7.4.4.

Lemma 7.4.3 *For all $d_t \geq 0$, $t = 1, \dots, T$, we have that*

$$\begin{aligned} & \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \right) = \\ & = \min_{t=1, \dots, T} \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right) \end{aligned}$$

where S is the unit simplex in \mathbb{R}^{mT} .

Proof: Note that

$$\begin{aligned} & \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \geq \\ & \geq \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \frac{1}{m} \sum_{i=1}^m \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \\ & = \sum_{t=1}^T \left(\sum_{i=1}^m \lambda_{it} \right) \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \frac{1}{m} \sum_{t=1}^T \left(\sum_{i=1}^m \lambda_{it} \right) \sum_{\tau=1}^t d_{\tau} \\ & = \sum_{t=1}^T \left(\sum_{i=1}^m \lambda_{it} \right) \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right) \\ & = \sum_{t=1}^T \mu_t \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right) \end{aligned}$$

where $\mu_t = \sum_{i=1}^m \lambda_{it}$, $\mu \in S'$, and S' is the unit simplex in \mathbb{R}^T . Thus,

$$\begin{aligned} & \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \right) \geq \\ & \geq \min_{\mu \in S'} \left(\sum_{t=1}^T \mu_t \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right) \right). \end{aligned}$$

On the other hand, for all $\mu \in S'$ we can define a vector $\lambda \in S$ through $\lambda_{it} = \frac{1}{m} \mu_t$ for each $i = 1, \dots, m$ and $t = 1, \dots, T$. We then have

$$\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} = \sum_{t=1}^T \mu_t \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right).$$

Thus,

$$\begin{aligned} & \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \right) \leq \\ & \leq \min_{\mu \in S'} \left(\sum_{t=1}^T \mu_t \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right) \right). \end{aligned}$$

This inequality together with the previous one turn into an equation. The minimization problem on μ is a linear programming problem and then its optimal value will be attained at an extreme point of S' which are equal to $\mu_{\tau}^{(\bar{\tau})} = 1$ if $\tau = \bar{\tau}$ and otherwise 0, for each $\bar{\tau} = 1, \dots, T$. Thus, the desired equation follows. \square

Theorem 7.4.4 *If $\mathcal{C} = \emptyset$ and all the facilities are identical, (P') is feasible with probability one, as $n \rightarrow \infty$, if*

$$m \cdot \sum_{\tau=1}^t \beta_{\tau} > \sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) \quad t = 1, \dots, T,$$

and infeasible with probability one if at least one of those inequalities is reversed.

Proof: First we will show the result for the static case, i.e., $\mathcal{S} = \{1, \dots, n\}$.

When all customers are static, the problem instances are asymptotically feasible with probability one if

$$\Delta = \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \right) > 0$$

(and infeasible with probability one if the inequality is reversed). Now let $\mathbf{D}_{1t} = d_t$ for $t = 1, \dots, T$ be realizations of the demands. Then, by Lemma 7.4.3, we have that

$$\min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \right) =$$

$$= \min_{t=1,\dots,T} \left(\sum_{\tau=1}^t \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^t d_{\tau} \right)$$

where $\beta_{\tau} = \beta_{i\tau}$ for each $i = 1, \dots, m$ and $\tau = 1, \dots, T$ (by hypothesis of the identical facilities model). For each $\bar{\tau} = 1, \dots, T$ define the set $S_{\bar{\tau}}$ so that $\lambda \in S_{\bar{\tau}}$ if

$$\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \min_{i=1,\dots,m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t d_{\tau} \geq \sum_{\tau=1}^{\bar{\tau}} \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^{\bar{\tau}} d_{\tau}.$$

Since the last inequality holds for all $\lambda \in S_{\bar{\tau}}$ and for all realizations d_t of the demands, we also have that

$$\begin{aligned} & \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \mathcal{E} \left(\min_{i=1,\dots,m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \geq \\ & \geq \sum_{\tau=1}^{\bar{\tau}} \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^{\bar{\tau}} \mathcal{E}(\mathbf{D}_{1\tau}) \end{aligned}$$

for all $\lambda \in S_{\bar{\tau}}$. But then

$$\begin{aligned} & \min_{\lambda \in S_{\bar{\tau}}} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \mathcal{E} \left(\min_{i=1,\dots,m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \right) \geq \\ & \geq \sum_{\tau=1}^{\bar{\tau}} \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^{\bar{\tau}} \mathcal{E}(\mathbf{D}_{1\tau}) \end{aligned}$$

and

$$\begin{aligned} & \min_{\bar{\tau}=1,\dots,T} \min_{\lambda \in S_{\bar{\tau}}} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \mathcal{E} \left(\min_{i=1,\dots,m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \right) \geq \\ & \geq \min_{\bar{\tau}=1,\dots,T} \left(\sum_{\tau=1}^{\bar{\tau}} \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^{\bar{\tau}} \mathcal{E}(\mathbf{D}_{1\tau}) \right). \end{aligned}$$

Noting that $S = \cup_{\bar{\tau}=1}^T S_{\bar{\tau}}$, this means that

$$\begin{aligned} & \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{\tau} \right) - \mathcal{E} \left(\min_{i=1,\dots,m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \mathbf{D}_{1\tau} \right) \right) \geq \\ & \geq \min_{\bar{\tau}=1,\dots,T} \left(\sum_{\tau=1}^{\bar{\tau}} \beta_{\tau} - \frac{1}{m} \sum_{\tau=1}^{\bar{\tau}} \mathcal{E}(\mathbf{D}_{1\tau}) \right). \end{aligned}$$

The reversed inequality can be obtained by considering the vectors $\lambda_{it} = \frac{1}{m}$ if $t = \bar{\tau}$ for each $i = 1, \dots, m$ and otherwise 0. Thus, the desired result follows for the static case.

Observe that the condition given in this theorem coincides with the condition found by Theorem 7.4.2 for the acyclic and dynamic case. Now the result also follows for the mixed case by using Lemma 7.4.1. \square

7.4.4 Seasonal demand pattern

In this section we analyze the special case of (P') where all customers have the same seasonal demand pattern, i.e., $d_{jt} = \sigma_t d_j$ for each $j = 1, \dots, n$ and $t = 1, \dots, T$ where σ_t are nonnegative constants. In the following result we obtain an explicit feasibility condition for the static case.

Theorem 7.4.5 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (P') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_\tau} \right) > \mathcal{E}(\mathbf{D}_1) \quad (7.6)$$

and infeasible with probability one if this inequality is reversed.

Proof: Note that the feasible region of (P') for the acyclic, static and seasonal case can be written as

$$\begin{aligned} \sum_{j=1}^n d_j x_{ij1} &\leq \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t b_{i\tau}}{\sum_{\tau=1}^t \sigma_\tau} \right) & i = 1, \dots, m \\ \sum_{i=1}^m x_{ij1} &= 1 & j = 1, \dots, n \\ x_{ij1} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n. \end{aligned}$$

Observe that this is the feasible region of a GAP with m agents and n tasks. The capacity of each agent is equal to $\min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t b_{i\tau}}{\sum_{\tau=1}^t \sigma_\tau} \right)$ and the requirement of each task is equal to d_j . Furthermore, the requirements are agent-independent. For that particular case, Romeijn and Piersma [110] derived the explicit feasibility condition saying that the expected requirement of a task must be below the total relative capacity which is the condition given in this theorem. \square

To illustrate the intuition behind this condition, consider the case where $\mathbf{D}_1 \equiv 1$ (that is, the *base demand* is not stochastic). Then, for a fixed i and large n ,

$$\min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_\tau} \right)$$

is equal to the fraction of the customers that can be assigned to facility i . Clearly, for feasibility the sum of those fractions over all facilities should be at least one.

When the capacity of each facility is invariant through the planning horizon, i.e. $b_{it} = b_i$ for all $t = 1, \dots, T$, the same condition holds for the mixed case.

Corollary 7.4.6 *If $\mathcal{C} = \emptyset$, $\beta_{it} = \beta_i$ for all $t = 1, \dots, T$, and all customers have the same demand pattern, (P') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\frac{\sum_{i=1}^m \beta_i}{\max_{t=1, \dots, T} \sum_{\tau=1}^t \sigma_\tau / t} > \mathcal{E}(\mathbf{D}_1) \quad (7.7)$$

and infeasible with probability one if this inequality is reversed.

Proof: We may observe that this condition coincides with the one obtained for the dynamic case in Theorem 7.4.2 and for the static and seasonal case in Theorem 7.4.5. The result follows now by using Lemma 7.4.1. \square

In contrary to the identical facilities case, for the seasonal case, the feasibility condition obtained when all customers are static is, in general, stronger than when all customers are dynamic. The following theorem proposes the solution of a linear programming problem to check whether the problem instances for the mixed case are feasible with probability one as n goes to infinity.

Theorem 7.4.7 *If $\mathcal{C} = \emptyset$ and all customers have the same demand pattern, (P') is feasible with probability one, as $n \rightarrow \infty$, if the optimal solution value of the following linear programming problem is strictly positive*

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \pi \mathcal{E}(\mathbf{D}_{11}) z_1 - (1 - \pi) \mathcal{E}(\mathbf{D}_{11}) \sum_{t=1}^T z_{2t}$$

subject to

$$\begin{aligned} z_1 &\leq \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \sigma_{\tau} & i = 1, \dots, m \\ z_{2t} &\leq \sum_{\tau=t}^T \lambda_{i\tau} & i = 1, \dots, m; t = 1, \dots, T \\ \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} &= 1 & i = 1, \dots, m; t = 1, \dots, T \\ \lambda_{it} &\geq 0 & i = 1, \dots, m; t = 1, \dots, T \\ z_1 &\text{free} \\ z_{2t} &\text{free} & t = 1, \dots, T, \end{aligned}$$

and infeasible with probability one if the optimal solution value is strictly negative.

Proof: For this particular case the excess capacity Δ reads as

$$\begin{aligned} \Delta &= \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \pi \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \sigma_{\tau} \mathbf{D}_{11} \right) \right. \\ &\quad \left. - (1 - \pi) \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \sigma_t \mathbf{D}_{11} \right) \right) \\ &= \min_{\lambda \in S} \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \pi \mathcal{E}(\mathbf{D}_{11}) \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \sigma_{\tau} \right) \right. \\ &\quad \left. - (1 - \pi) \mathcal{E}(\mathbf{D}_{11}) \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \sigma_t \right) \right) \end{aligned}$$

where S is the unit simplex in \mathbb{R}^{mT} . We can verify if $\Delta > 0$ holds by solving the following mathematical program:

$$\begin{aligned} \text{minimize} \quad & \left(\sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \pi \mathcal{E}(\mathbf{D}_{11}) \left(\min_{i=1, \dots, m} \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \sigma_{\tau} \right) \right. \\ & \left. - (1 - \pi) \mathcal{E}(\mathbf{D}_{11}) \sum_{t=1}^T \mathcal{E} \left(\min_{i=1, \dots, m} \sum_{\tau=t}^T \lambda_{i\tau} \sigma_t \right) \right) \end{aligned}$$

subject to

$$\lambda \in S.$$

It is easy to see that this problem can be rewritten as

$$\text{minimize} \quad \sum_{t=1}^T \sum_{i=1}^m \lambda_{it} \left(\sum_{\tau=1}^t \beta_{i\tau} \right) - \pi \mathcal{E}(\mathbf{D}_{11}) z_1 - (1 - \pi) \mathcal{E}(\mathbf{D}_{11}) \sum_{t=1}^T z_{2t}$$

subject to

$$\begin{aligned} z_1 &\leq \sum_{t=1}^T \lambda_{it} \sum_{\tau=1}^t \sigma_{\tau} & i = 1, \dots, m \\ z_{2t} &\leq \sum_{\tau=t}^T \lambda_{i\tau} & i = 1, \dots, m; \ t = 1, \dots, T \\ \lambda &\in S \\ z_1 &\text{ free} \\ z_{2t} &\text{ free} & t = 1, \dots, T. \end{aligned}$$

Note that S is a polyhedral set, and thus the program is a linear programming problem. \square

7.5 Numerical illustrations

In this section we will numerically illustrate some of the results of this chapter. We have considered a collection of classes of problem instances for the MPSSP. The variety is based on the type of facility and the *limit* ratio between static and dynamic customers, say π . More precisely, we have considered the pure cyclic and the pure acyclic cases. For both cases, we have generated problem instances with only static customers ($\pi = 1$), problem instances with only dynamic customers ($\pi = 0$), and a mixed case ($\pi = 0.5$).

We have generated for each customer a random demand \mathbf{D}_{jt} in period t from the uniform distribution on $[5\alpha_t, 25\alpha_t]$. We have chosen the vector of seasonal factors to

be $\alpha = (\frac{1}{2}, \frac{3}{4}, 1, 1, \frac{3}{4}, \frac{1}{2})^\top$. We have chosen the capacities $b_{it} = \beta \cdot n$. For this special case, the condition for feasibility reduces to

$$\beta > \frac{15}{m} \cdot \frac{1}{T} \sum_{t=1}^T \alpha_t \equiv \beta_{\min}^c$$

for the cyclic case, and

$$\beta > \frac{15}{m} \cdot \max_{\tau=1, \dots, T} \left(\frac{1}{\tau} \sum_{t=1}^{\tau} \alpha_t \right) \equiv \beta_{\min}^a$$

for the acyclic case.

We have fixed the number of facilities at $m = 5$, and the number of periods at $T = 6$. The number of customers ranges from $n = 10$ until $n = 100$ with incremental steps of 5. For each class of problem instances and each size of the problem we have generated 50 problem instances. All problem instances were solved using the MIP solver of CPLEX 6.5 [33]. The optimization procedure was stopped as soon as a feasible solution was found or infeasibility could be proved.

Figures 7.2–7.4 show, for various values of β , the fraction of feasible problem instances generated for the cyclic case when $\pi = 1, 0.5$ and 0 . Due to the hardness of the MPSSP, we allowed a maximal computational time of 30 minutes. Moreover, CPLEX also imposes a limitation on the used memory. Therefore, the fraction of feasible instances was calculated based on the problem instances for which a feasible solution was found or it could be proved that the problem was infeasible. Similar results are illustrated for the acyclic case, see Figures 7.5–7.7.

As expected, the fraction of feasible problem instances is close to one when the number of customers n grows if the relative capacity is 5% over the theoretical lower bound. On the other hand, this fraction is close to zero when the relative capacity is 5% below the theoretical lower bound. Recall that we were not able to find theoretical results when the relative capacity is equal to the theoretical lower bound. From those figures, it looks like the fraction of feasible problem instances is around 0.5 when n grows. We also observe almost the same the pattern for the fraction of feasible problem instances for the different classes of multi-period single-sourcing problems generated in this section.

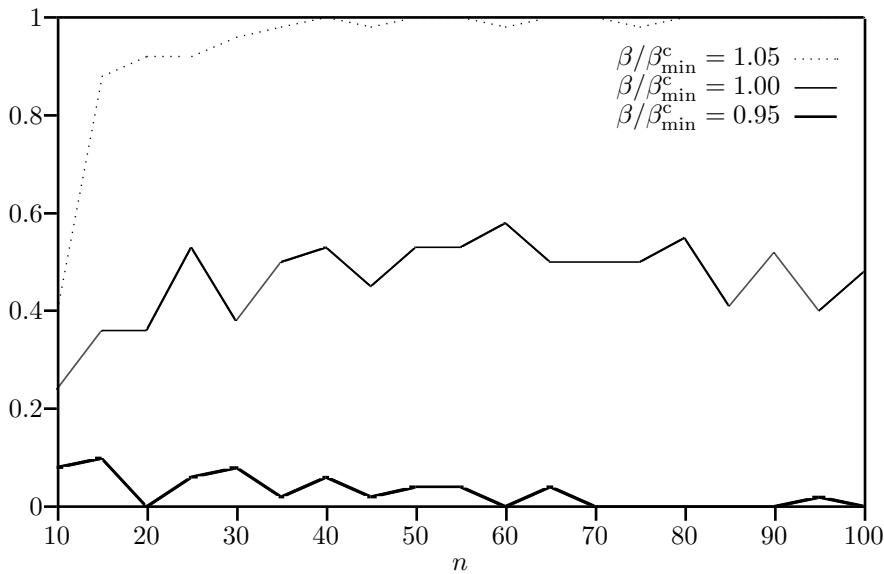


Figure 7.2: Average feasibility, cyclic and static

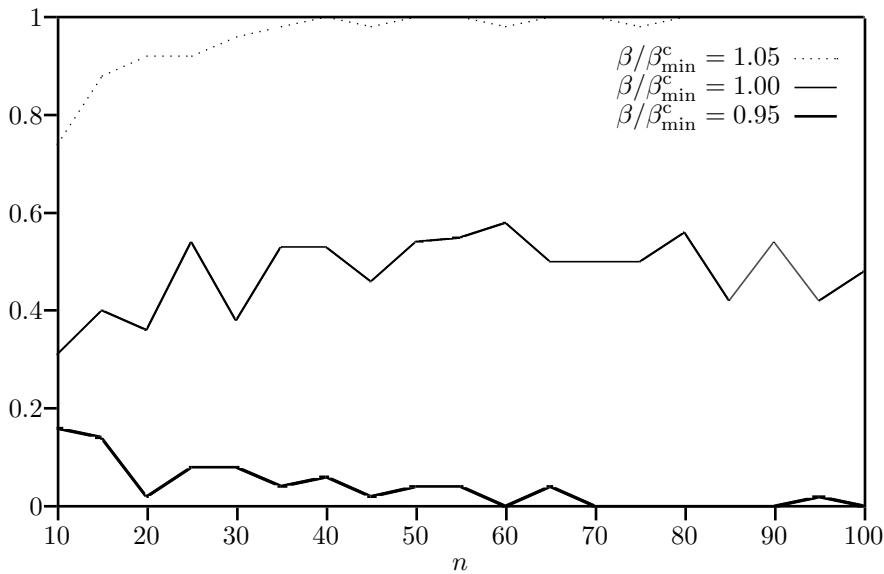


Figure 7.3: Average feasibility, cyclic and mixed

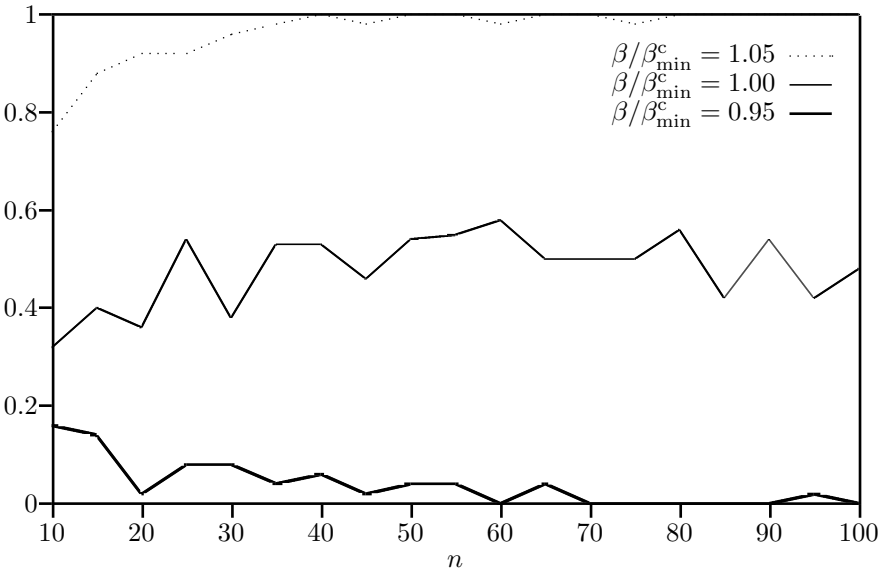


Figure 7.4: Average feasibility, cyclic and dynamic

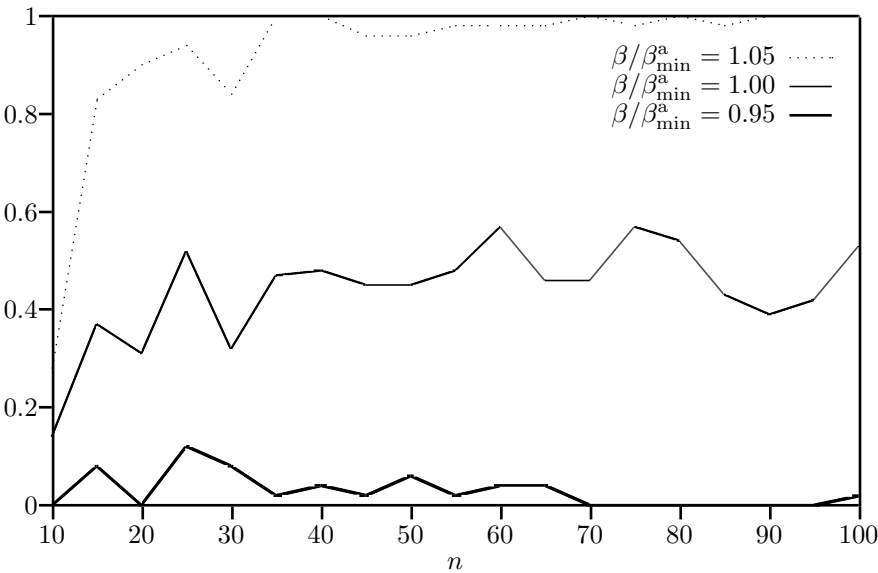


Figure 7.5: Average feasibility, acyclic and static

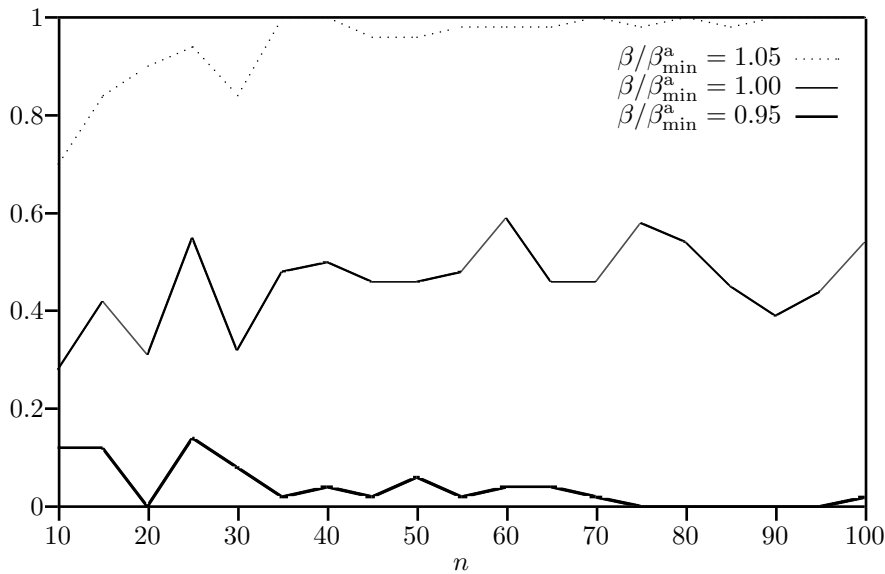


Figure 7.6: Average feasibility, acyclic and mixed

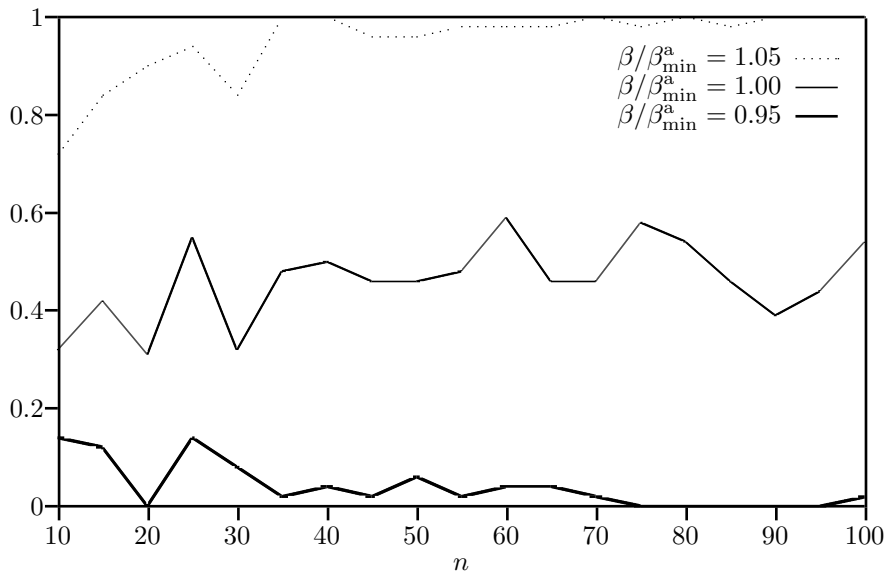


Figure 7.7: Average feasibility, acyclic and dynamic

Chapter 8

Asymptotical analysis of a greedy heuristic for the MPSSP

8.1 Introduction

The possibility of opportunities to improve the performance of the logistics distribution network leads to management conducting an analysis to detect (when existing) these opportunities, and subsequently, take advantage of them. A mathematical model is a powerful tool to represent the situation to be analyzed. It is obvious that this model is not able to catch all the practical details. Often, management proposes several scenarios which can be closely represented by the mathematical model. For example, the MPSSP evaluates the total production, transportation, handling and inventory costs of a determined layout of the logistics distribution network. The parameters used by the mathematical model are frequently estimates. Management may require to investigate the revision of some of those parameters. The delivery costs are a clear example for the MPSSP. These costs depend on the warehouse dealing with the customer, the frequency of delivery, the routes on which the customer is served, etc. If the route consists of a direct shipment to the customer then we can easily find an expression for them. However, the demands of several customers are usually combined in a single route. Those routes are designed on a daily basis and therefore unknown at a strategical or tactical level.

From the discussion above it follows that solutions to several problem instances of the MPSSP may be required before management can take a decision. The dimension of the MPSSP grows linearly in the number of customers which, in practice, is large. Hence, the \mathcal{NP} -Hardness of the MPSSP (see Chapter 6) implies that supporting management to take a decision can be computationally expensive. Generally, a good feasible solution for the MPSSP satisfies the needs of the managers. Moreover, a good starting feasible solution can save quite some effort when solving the MPSSP

by means of an exact solution procedure. In this chapter we are interested in finding heuristic solution approaches yielding good feasible solutions for the MPSSP.

A formulation as a convex capacitated assignment problem was given for the MPSSP in Chapter 6. Recall that in the CCAP we have a set of tasks and a set of agents which can process those tasks. Each task has to be processed by exactly one agent, and capacity constraints are faced by the agents when processing the tasks. In Chapter 2 we have proposed a class of greedy heuristics for the CCAP, each one requiring a pseudo-cost function evaluating the assignment of tasks to agents. In this chapter we propose a family of pseudo-cost functions for the MPSSP based on the results experienced for the GAP in Chapter 5. As for the GAP, we will pay special attention to a member of this family obtained by using some information of the LP-relaxation of the MPSSP. We will prove asymptotic feasibility and optimality of this greedy heuristic for the cyclic MPSSP. We are able to prove similar results for two large classes of problem instances of the acyclic MPSSP.

The formulation of the MPSSP as a CCAP, (P'), is, in general, nonlinear. We have also proposed in Chapter 6 a linear formulation which has certain advantages when analyzing the asymptotic behaviour of the abovementioned greedy heuristic for the MPSSP. Recall that the LP-relaxation of the MPSSP, (LPR), and its dual programming problem, (D), were formulated in Section 6.4. The vector x^{LPR} denotes the optimal assignments for (LPR). The equivalence shown for the two formulations of the MPSSP also holds for their relaxations. Hence, x^{LPR} is also the optimal solution for the relaxation of the nonlinear formulation (P'). Finally, recall that $B_S \cup B_D$ denotes the set of split assignments of x^{LPR} , and x^G the (partial) solution for the MPSSP given by the greedy heuristic.

The outline of this chapter is as follows. In Section 8.2 we will introduce a family of pseudo-cost functions for the greedy heuristic for the CCAP for the particular case of the MPSSP each one requiring a vector of nonnegative multipliers. Throughout this chapter we will analyze the asymptotic behaviour of the member of this family for which the vector of multipliers is equal to the optimal dual subvector corresponding to the capacity constraints in (LPR). In Section 8.3 we propose a general stochastic model for the parameters of the MPSSP under which we analyze the behaviour of the greedy heuristic. In Section 8.4 we prove asymptotic feasibility and optimality in the probabilistic sense of the greedy heuristic on problem instances of the MPSSP when all facilities show a cyclic inventory pattern. In Section 8.4 we analyze the acyclic MPSSP. We will show asymptotic feasibility and optimality in a probabilistic sense for two large subclasses of problems, namely the acyclic MPSSP with dynamic customers and the acyclic MPSSP with static customers where each customer's demand pattern exhibits the same seasonality pattern. Finally, Section 8.6 presents some numerical results to illustrate the behaviour of the greedy heuristic. Some of the results in this chapter can be found in Romeijn and Romero Morales [113, 114].

8.2 A family of pseudo-cost functions

In the introduction we have mentioned that the MPSSP can be formulated as a CCAP. The class of convex capacitated assignment problems was extensively studied

in Chapter 2. Due to the hardness of this problem, one of the main points of study was the development of greedy heuristic solution procedures for the CCAP. We proposed a class of greedy heuristics for the CCAP based on the definition of a pseudo-cost function measuring the assignment of tasks to agents. This type of greedy heuristics was used in Chapter 5 to solve the GAP. We defined in that chapter a family of pseudo-cost functions. Special attention was paid to two members of this family for which, under additional assumptions, asymptotic feasibility and optimality was proved. In this section we define a pseudo-cost function for the MPSSP inspired in the results found for the GAP.

The greedy heuristic for the CCAP basically works as follows. The assignment of task ℓ to agent i is evaluated by a pseudo-cost function $f(i, \ell)$. By means of this pseudo-cost function we measure the desirability of assigning task ℓ as the difference between the second smallest and the smallest values of $f(i, \ell)$ over the set of agents. We create a list of the tasks in which they appear in decreasing order of their desirability. Tasks are assigned to their best agent according to this list. Along the way, some agents will not be able anymore to deal with some of the tasks because of the capacity constraints they face. Therefore, the desirabilities must be updated taking into account that the two most desirable agents for each task should be feasible.

Recall that each static customer induces exactly a task in the CCAP formulation of the MPSSP, while a dynamic one induces T tasks representing the assignment of this customer in period t , for each $t = 1, \dots, T$. Similarly as for the GAP in Chapter 5, we consider the family of pseudo-cost functions given by

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T (c_{ijt} + \lambda_{it} d_{jt}) & \text{if } \ell = j \in \mathcal{S} \\ c_{ijt} + \lambda_{it} d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda \in \mathbb{R}_+^{mT}$. Throughout this chapter we will analyze the asymptotic behaviour of the greedy heuristic with $\lambda = \lambda^*$, where $\lambda^* = (\lambda_{it}^*)$ represents the optimal subvector to (D) corresponding to the capacity constraints of (LPR) when (LPR) is feasible and an arbitrary nonnegative vector when (LPR) is infeasible. (Clearly, if (LPR) is infeasible, so is the MPSSP.) Observe that the capacity constraints has been reformulated as \geq -constraints, so that their dual multipliers are nonnegative.

8.3 A probabilistic model

In the following two sections we will probabilistically analyze the asymptotic behaviour of the greedy heuristic for the MPSSP introduced in the previous section. Recall that we have proposed a stochastic model for the parameters defining the feasible region of the MPSSP in Chapter 7. Since we allow for dependencies between costs and requirements parameters, we need to redefine the stochastic model rather than simply add distribution assumptions on the costs parameters. Let the random vectors $(\mathbf{D}_j, \mathbf{C}_j, \gamma_j)$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $[\underline{D}, \overline{D}]^T \times [\underline{C}, \overline{C}]^{mT} \times \{0, 1\}$ where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, $\mathbf{C}_j = (\mathbf{C}_{ijt})_{i=1, \dots, m; t=1, \dots, T}$, $(\mathbf{D}_j, \mathbf{C}_j)$ are distributed according to an absolutely continuous probability distribution and $\underline{D}, \overline{D}, \underline{C}$ and $\overline{C} \in \mathbb{R}_+$. Furthermore, let b_{it} depend linearly on n , i.e., $b_{it} = \beta_{it}n$, for positive

constants $\beta_{it} \in \mathbb{R}_+$. Throughout this chapter we will assume that $\underline{D} > 0$. Note that, except for $\underline{D} > 0$, the conditions on the requirement parameters will be the same as for the earlier stochastic model. In Chapter 7 we have shown that the following assumption ensures asymptotic feasibility of the MPSSP with probability one.

Assumption 8.3.1 *Assume that the excess capacity Δ defined as in (7.1) is strictly positive.*

Throughout this chapter we will suppose that Assumption 8.3.1 holds to ensure that, when n grows to infinity, we generate feasible problem instances for the MPSSP with probability one. In Chapter 7, we have found more explicit conditions equivalent to $\Delta > 0$ for subclasses of MPSSP's which will be stated and used in Sections 8.4 and 8.5.

The following proposition ensures that (LPR) is non-degenerate with probability one.

Proposition 8.3.2 *(LPR) is non-degenerate with probability one, under the proposed stochastic model.*

Proof: This follows directly from the fact that, for each customer, the vector of cost and requirement parameters is distributed according to an absolutely continuous probability distribution. \square

In the remainder, let \mathbf{X}_n^G denote the random vector representing the (partial) solution for the MPSSP given by the greedy heuristic, and Z_n^G be its objective value. Let $\mathbf{X}_n^{\text{LPR}}$ be the random vector representing the optimal assignments for (LPR), and Z_n^{LPR} be the optimal objective value of (LPR).

Recall that \mathcal{C} denotes the set of facilities where the inventory pattern is cyclic. As mentioned in Chapter 6, the only realistic cases for the MPSSP are when all facilities are of the same type, i.e., all facilities are cyclic ($\mathcal{C} = \{1, \dots, m\}$) or all are acyclic ($\mathcal{C} = \emptyset$). In the following, we will analyze separately the two cases and we will skip the mixed one.

8.4 An asymptotically optimal greedy heuristic: the cyclic case

In this section we consider the cyclic multi-period single-sourcing problem, i.e., when $\mathcal{C} = \{1, \dots, m\}$. We have already remarked in Chapter 7 that the reformulation of the cyclic MPSSP as a CCAP yields a GAP with a nonlinear function. In particular, the objective function reads as the sum of a linear term and a Lipschitz function (see Proposition 6.3.4). As for the GAP (see Section 5.4.2), the key result to show the asymptotic feasibility and optimality of the greedy heuristic is to prove that the number of assignments on which the greedy heuristic and (LPR) differ can be bounded by an expression independent of n . Therefore, their values are also close since the objective function of the MPSSP is Lipschitzian.

As showed in Theorem 7.3.1, Assumption 8.3.1, ensuring asymptotic feasibility of the problem instances generated by the stochastic model for the MPSSP, reads as follows when all facilities are cyclic.

Assumption 8.4.1 *Assume that*

$$\sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) < \sum_{t=1}^T \sum_{i=1}^m \beta_{it}.$$

To show the asymptotic feasibility of the greedy heuristic, we first prove that, as n grows to infinity, the aggregate slack in the optimal LP-solution grows linearly in n .

Lemma 8.4.2 *It holds that*

$$\sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \frac{1}{n} \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n \mathbf{D}_{jt} \mathbf{X}_{ijt}^{\text{LPR}} > 0$$

with probability one when n goes to infinity.

Proof: Note that

$$\begin{aligned} & \sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \frac{1}{n} \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n d_{jt} x_{ijt}^{\text{LPR}} = \\ &= \sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n d_{jt} \left(\sum_{i=1}^m x_{ijt}^{\text{LPR}} \right) \\ &= \sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n d_{jt}. \end{aligned}$$

We have that

$$\begin{aligned} & \sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \frac{1}{n} \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n \mathbf{D}_{jt} \mathbf{X}_{ijt}^{\text{LPR}} = \\ &= \sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \frac{1}{n} \sum_{t=1}^T \sum_{j=1}^n \mathbf{D}_{jt} \\ &\rightarrow \sum_{t=1}^T \sum_{i=1}^m \beta_{it} - \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) \\ &> 0 \end{aligned}$$

with probability one as n goes to infinity, by the Law of the Large Numbers and Assumption 8.4.1. \square

Theorem 8.4.3 shows that the (partial) solution found by the greedy heuristic using the pseudo-cost function proposed in Section 8.2 and the optimal vector of assignments for (LPR) coincide for almost all assignments that are feasible in the latter. Let N_n be the set of assignments which do not coincide in x^G and in x^{LPR} , i.e.,

$$\begin{aligned} N_n = & \{j \in \mathcal{S} : \exists i = 1, \dots, m \text{ such that } x_{ij1}^G \neq x_{ij1}^{\text{LPR}}\} \\ & \cup \{(j, t) : j \in \mathcal{D}, \exists i = 1, \dots, m \text{ such that } x_{ijt}^G \neq x_{ijt}^{\text{LPR}}\}. \end{aligned}$$

(Note that, for static customers, we only count the assignment made in period 1, since the assignments in the other periods are necessarily equal to that assignment.) As for the GAP, we will prove that $|N_n|$ is bounded by an expression independent of n .

Theorem 8.4.3 *There exists a constant R , independent of n , so that $|N_n| \leq R$ for all problem instances of (LPR) that are feasible and non-degenerate.*

Proof: Recall that when all facilities show a cyclic inventory pattern, the MPSSP can be reformulated as a GAP with m agents, $|\mathcal{S}| + T \cdot |\mathcal{D}|$ tasks, and a nonlinear objective function. In this reformulation, the facilities can be seen as the agents while the tasks are associated with the assignment of the customers. Each static customer j defines a task requiring $\sum_{t=1}^T d_{jt}$ units of capacity. Each dynamic customer j brings T new tasks each one associated with the assignment of that customer in period t and requiring d_{jt} units of capacity, for each $t = 1, \dots, T$. Therefore, the minimal capacity required by a task is lower bounded by \underline{D} , and the maximal one is upper bounded by $T\bar{D}$.

As for the GAP (see Theorem 5.4.8), the most desirable agent for each task that is feasibly assigned in x^{LPR} is equal to the agent to which is assigned in x^{LPR} (see Proposition 6.4.3). Moreover, the same proposition shows that the initial desirabilities are such that the greedy heuristic starts by assigning tasks that are feasibly assigned in x^{LPR} . Now suppose that the greedy heuristic would reproduce all the assignments that are feasible in x^{LPR} . Then, because the remaining assignments in x^{LPR} are infeasible with respect to the integrality constraints, x^G and x^{LPR} would differ only in those last ones. We know that then $|N_n| = |B_S| + |B_D| \leq mT$ where this inequality follows from Lemma 6.4.1. Thus, $|N_n|$ is bounded from above by a constant independent of n , and the result holds.

So it remains to prove the result when x^G and x^{LPR} differ in at least one assignment that is feasible in the latter. The proof developed for the (linear) GAP also holds here, since the linearity of the objective function was not used. Therefore, we can derive that

$$|N_n| \leq mT \left(2 + \left\lceil \frac{T\bar{D}}{\underline{D}} \right\rceil \right)^{m \left\lceil \frac{T\bar{D}}{\underline{D}} \right\rceil - 1}$$

where mT represents an upper bound on the number of infeasible assignments in (LPR) with respect to the integrality constraints, $T\bar{D}$ is an upper bound on the maximal capacity required by a task, \underline{D} a lower bound on the minimal capacity, and

m in the exponent is the number of agents in the GAP formulation of the cyclic MPSSP. Again $|N_n|$ is bounded from above by a constant independent of n , and the desired result follows. \square

In Theorem 8.4.4 we state that the greedy heuristic given in Section 8.2 is asymptotically feasible with probability one when all facilities are cyclic. This proof combines the results of Theorem 8.4.3, where it is shown that x^{LPR} and x^{G} coincide for almost all the feasible assignments in x^{LPR} , and Theorem 8.4.2.

Theorem 8.4.4 *If $\mathcal{C} = \{1, \dots, m\}$, the greedy heuristic given in Section 8.2 is asymptotically feasible with probability one.*

Proof: The result follows similarly as Theorem 5.4.9. From Theorem 8.4.3 we know that the number of assignments that differ between the optimal vector of assignments of (LPR) and the solution given by the greedy heuristic is bounded by a constant independent of n . Moreover, Lemma 8.4.2 ensures us that the aggregate remaining capacity in the optimal vector of assignments for (LPR) grows linearly with n . Thus, when n grows to infinity, there is enough available capacity to fix the remaining assignments. \square

In Theorem 8.4.5 we show that the greedy heuristic is asymptotically optimal with probability one.

Theorem 8.4.5 *If $\mathcal{C} = \{1, \dots, m\}$, the greedy heuristic given in Section 8.2 is asymptotically optimal with probability one.*

Proof: From Theorem 8.4.4 we know that the greedy heuristic is asymptotically feasible with probability one.

It thus suffices to show that $\left(\frac{1}{n}Z_n^{\text{G}} - \frac{1}{n}Z_n^{\text{LPR}}\right) \rightarrow 0$ with probability one. By definition, we have that

$$\begin{aligned}
 & \frac{1}{n}Z_n^{\text{G}} - \frac{1}{n}Z_n^{\text{LPR}} = \\
 &= \frac{1}{n} \left(\sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n C_{ijt} \mathbf{X}_{ijt}^{\text{G}} + \sum_{i=1}^m H_i(\mathbf{X}_{i..}^{\text{G}}) \right) \\
 & \quad - \frac{1}{n} \left(\sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n C_{ijt} \mathbf{X}_{ijt}^{\text{LPR}} + \sum_{i=1}^m H_i(\mathbf{X}_{i..}^{\text{LPR}}) \right) \\
 &\leq (\bar{C} - \underline{C}) \frac{1}{n} \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n (\mathbf{X}_{ijt}^{\text{G}} - \mathbf{X}_{ijt}^{\text{LPR}}) \\
 & \quad + \frac{1}{n} \left| \sum_{i=1}^m H_i(\mathbf{X}_{i..}^{\text{G}}) - \sum_{i=1}^m H_i(\mathbf{X}_{i..}^{\text{LPR}}) \right| \\
 &\leq (\bar{C} - \underline{C}) \frac{1}{n} \left\| \mathbf{X}^{\text{G}} - \mathbf{X}^{\text{LPR}} \right\|_1 + \frac{1}{n} L \left\| \mathbf{X}^{\text{G}} - \mathbf{X}^{\text{LPR}} \right\|_1 \tag{8.1}
 \end{aligned}$$

$$\leq (\bar{C} - \underline{C} + L) \cdot mT \frac{|N_n|}{n} \quad (8.2)$$

where inequality (8.1) follows by Proposition 6.3.4, L is defined by (6.17), and inequality (8.2) follows by the definition of set N_n . The result then follows from Theorem 8.4.3. \square

8.5 Asymptotic analysis: the acyclic case

8.5.1 Introduction

In the previous section we have proved asymptotic feasibility and optimality of the greedy heuristic given in Section 8.2 in the probabilistic sense when all facilities have a cyclic inventory pattern. This section is devoted to the analysis of the same greedy heuristic when all facilities are acyclic. In opposite to the cyclic case, the reformulation of the MPSSP as a CCAP yields, in general, a *real* dynamic model. We have not been able to prove asymptotic optimality for all cases. As for the feasibility analysis of the MPSSP performed in Chapter 7, the fact that the static customers have to be assigned to a single facility throughout the planning horizon makes the problem a truly dynamic model, avoiding any sequential analysis of the problem. We will be still able to prove asymptotic results when all customers are dynamic and a large subclass of problems of the static case.

As in Section 8.4, let N_n define the set of assignments where x^G and x^{LPR} differ. We will show again that the cardinality of this set is bounded by an expression independent of n .

Theorem 8.5.1 *There exists a constant R , independent of n , so that $|N_n| \leq R$ for all problem instances of (LPR) that are feasible and non-degenerate.*

Proof: The acyclic MPSSP can be reformulated as a CCAP with the same type of agents and tasks as in the cyclic case, i.e., each facility defines an agent, each static customer a task and each dynamic customer T tasks. While the cyclic agents face only a capacity constraint, the acyclic ones confront with T of them. The t -th capacity constraint restricts the aggregate consumption over the first t periods, for each $t = 1, \dots, T$.

A similar proof as for the GAP (see Theorem 5.4.8) applies here. Again we can show that the most desirable agent for each task that is feasibly assigned in x^{LPR} is equal to the agent to which it is assigned in x^{LPR} . Moreover, the initial desirabilities are such that the greedy heuristic starts by assigning tasks that are feasibly assigned in x^{LPR} . If the greedy heuristic would reproduce all the assignments that are feasible in x^{LPR} , then $|N_n| \leq mT$. Thus, $|N_n|$ is bounded from above by a constant independent of n , and the result follows. So in the remainder of the proof we will assume that x^G and x^{LPR} differ in at least one assignment that is feasible in the latter. As for the GAP, it suffices to bound the number of times the desirabilities must be recalculated, and the number of affected assignments by an assignment in which x^G and x^{LPR} differ.

While fixing assignments the remaining aggregate capacities at the agents decrease, and the greedy heuristic may need to update the desirabilities. This may cause the greedy heuristic to deviate from one of the feasible assignments in x^{LPR} which in turn can cause additional deviations. In particular, for each static customer, this assignment uses at most $t\bar{D}$ units of capacity through period t , for each $t = 1, \dots, T$, while for each dynamic one at most \bar{D} . Since any other static assignment requires at least $t\underline{D}$ units of capacity through period t , for each $t = 1, \dots, T$, and each dynamic one at least \underline{D} , the number of additional deviations is at most equal to $\max_{t=1, \dots, T} \lceil (t\bar{D})/\underline{D} \rceil = \lceil T\bar{D}/\underline{D} \rceil$.

The calculation of the desirabilities depends only on the set of feasible agents for each task which was not assigned yet. The feasibility of an agent is a potential issue, corresponding to a need to recalculate the values of the desirabilities, only when its cumulative available capacity through period t is below $t\bar{D}$ for some t . As above, a static assignment uses at least $t\underline{D}$ units of cumulative capacity in each period t , while a dynamic assignment uses at least \underline{D} units of capacity in at least one period. Thus, the number of times that the desirabilities must be recalculated is at most $mT \lceil \frac{T\bar{D}}{\underline{D}} \rceil$.

Now following a similar recursion as in the proof of Theorem 5.4.8, we have that

$$|N_n| \leq mT \left(2 + \left\lceil \frac{T\bar{D}}{\underline{D}} \right\rceil \right)^{mT \lceil \frac{T\bar{D}}{\underline{D}} \rceil - 1}.$$

Thus, $|N_n|$ is bounded from above by a constant independent of n , and the result follows. \square

8.5.2 Only dynamic customers

In this section we study the asymptotic behaviour of the greedy heuristic presented in Section 8.2 for the acyclic MPSSP when all customers are dynamic. In opposite to the cyclic MPSSP, this is a *real* dynamic model which complicates its analysis. Two results were key to the proof of asymptotic feasibility and optimality of the greedy heuristic for the cyclic case. On one hand, we have proved that the number of assignments in which the greedy heuristic and (LPR) differ can be bounded by an expression independent of n . On the other hand, we have shown that the aggregate slack in the optimal LP-solution grows linearly in n . We can show that those two results hold in the dynamic case as well. However, since capacity can only be transferred to future periods in the acyclic MPSSP, those results are not enough to show asymptotic feasibility. Because of the linear growth on n of the aggregate slack in the optimal LP-solution, we know that there is enough remaining capacity, but it could be available too late in time. We can remedy this with a sequential procedure which improves the feasibility of x^G .

As showed in Theorem 7.4.2, Assumption 8.3.1 reads as follows when all facilities are acyclic and all customers are dynamic.

Assumption 8.5.2 *Assume that*

$$\sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) < \sum_{\tau=1}^t \sum_{i=1}^m \beta_{i\tau} \quad \text{for } t = 1, \dots, T.$$

Recall that when all customers are dynamic the tasks can be seen as the (customer, period)-pairs. After the greedy heuristic has been applied, some of the tasks may have not been assigned, i.e., $\mathcal{NA} \neq \emptyset$, see Section 2.3.1. We describe a sequential procedure to try to assign the tasks in \mathcal{NA} . Associated with x^G there exists a vector $I^G \in \mathbb{R}^{mT}$ so that (x^G, I^G) is a feasible partial solution for the MPSSP. After fixing the assignments from the greedy heuristic the remaining aggregate capacities are equal to

$$\sum_{\tau=1}^t b_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{ij\tau}^G - I_{it}^G$$

for each $i = 1, \dots, m$ and $t = 1, \dots, T$. The elements of the set \mathcal{NA} will be considered in increasing order of the period to which they belong. Suppose that we are in a particular period t and pair (j, t) is the next element of \mathcal{NA} under consideration. If there exists a facility i with remaining capacity in periods $\tau = t, \dots, T$ at least d_{jt} , then we can assign (j, t) to facility i . Otherwise, if there exists a facility i where

$$\sum_{\tau=1}^t b_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{ij\tau}^G > d_{jt} \quad (8.3)$$

then we may still be able to assign (j, t) to facility i by reversing some assignments in later periods, and adding those to the set \mathcal{NA} . We will then proceed with the next element of \mathcal{NA} . If there is no facility for which equation (8.3) holds, then the sequential procedure is not able to assign (j, t) . We will show that this occurs with probability zero as n goes to infinity.

Sequential procedure

Step 0. Let x^G be the current partial solution for the MPSSP. Set $\mathcal{I} = \emptyset$ and $t = 1$.

Step 1. If $\{j : (j, t) \in \mathcal{NA}\} = \emptyset$, go to Step 4. Otherwise, choose

$$\hat{j} = \arg \max_{j:(j,t) \in \mathcal{NA}} d_{jt}$$

(where ties are broken arbitrarily).

Step 2. If there exists some facility \hat{i} so that

$$\sum_{\tau=1}^{\bar{\tau}} b_{i\tau} - \sum_{\tau=1}^{\bar{\tau}} \sum_{j=1}^n d_{j\tau} x_{ij\tau}^G > d_{\hat{j}t} \quad \text{for } \bar{\tau} = t, \dots, T,$$

then set

$$\begin{aligned} x_{i\hat{j}t}^G &= 1 \\ \mathcal{NA} &= \mathcal{NA} \setminus \{(\hat{j}, t)\} \end{aligned}$$

and go to Step 1.

Step 3. If there exists some facility \hat{i} so that

$$\sum_{\tau=1}^t b_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{i\hat{j}\tau}^G > d_{\hat{j}t},$$

then find a collection of pairs $(j_1, t_1), \dots, (j_s, t_s)$ so that $x_{i\hat{j}_k t_k}^G = 1$ where $t_k > t$ for each $k = 1, \dots, s$, such that reversing the assignments in this collection makes the assignment of (\hat{j}, t) to \hat{i} feasible. Then, set

$$\begin{aligned} x_{i\hat{j}t}^G &= 1 \\ x_{i\hat{j}_k t_k}^G &= 0 \quad \text{for } k = 1, \dots, s \\ \mathcal{NA} &= \left(\mathcal{NA} \cup \bigcup_{k=1}^s \{(j_k, t_k)\} \right) \setminus \{(\hat{j}, t)\} \end{aligned}$$

and go to Step 1. If such a facility does not exist, set

$$\begin{aligned} \mathcal{I} &= \mathcal{I} \cup \{(\hat{j}, t)\} \\ \mathcal{NA} &= \mathcal{NA} \setminus \{(\hat{j}, t)\} \end{aligned}$$

and go to Step 1.

Step 4. If $t = T$, STOP: x^G is a feasible solution for the MPSSP if $\mathcal{I} = \emptyset$, and otherwise is a partial solution. If $t < T$, increment t by one and go to Step 1.

In Theorem 8.5.3 we show that the greedy heuristic given in Section 8.2 followed by the sequential procedure is asymptotically feasible with probability one.

Theorem 8.5.3 *If $\mathcal{C} = \emptyset$ and $\mathcal{S} = \emptyset$, the greedy heuristic given in Section 8.2 combined with the sequential procedure are asymptotically feasible with probability one.*

Proof: It suffices to show that the sequential procedure applied to the partial solution obtained by the greedy heuristic, x^G , is asymptotically feasible. In particular, we will derive a set of sufficient conditions under which the sequential procedure applied to x^G will find a feasible solution. We will then show that this set of conditions is satisfied with probability one when n goes to infinity.

Recall that the sequential procedure considers the pairs in \mathcal{NA} for assignment in increasing order of the period to which they belong. Now let \mathcal{NA}_t be the set of

unassigned pairs after the sequential procedure has considered all pairs from periods 1 until $t - 1$ ($t = 2, \dots, T$), and define $\mathcal{NA}_1 \equiv \mathcal{NA}$. Consider some period t ($t = 1, \dots, T$). To assign any pair in that period, it is easy to see that we have to unassign at most $\lceil \overline{D}/\underline{D} \rceil$ pairs in future periods. So each element of \mathcal{NA}_t that gets assigned in period t yields at most $\lceil \overline{D}/\underline{D} \rceil$ pairs in future periods that need to be assigned, and each element of \mathcal{NA}_t that does not correspond to period t simply remains to be assigned in a future period. This implies that

$$|\mathcal{NA}_{t+1}| \leq \lceil \overline{D}/\underline{D} \rceil |\mathcal{NA}_t|$$

for $t = 1, \dots, T$. Using the fact that $|\mathcal{NA}| \leq |N_n|$, it is easy to see that $|\mathcal{NA}_t|$ is bounded from above by a constant independent of n (see Theorem 8.5.1).

Now consider the first period. The set of pairs that remain to be assigned to a facility in this period is equal to \mathcal{NA}_1 . Recall that the sequential procedure is able to assign a pair $(j, 1) \in \mathcal{NA}_1$ to facility i if

$$b_{i1} - \sum_{j=1}^n d_{j1} x_{ij1}^G > d_{j1}.$$

Such a facility exists for all customers that remain to be assigned in period 1 if

$$\sum_{i=1}^m b_{i1} - \sum_{i=1}^m \sum_{j=1}^n d_{j1} x_{ij1}^G > m\overline{D} \max\{2, |\mathcal{NA}_1|\} \equiv K_1.$$

Similarly, it can be shown for $t = 2, \dots, T$ that

$$\sum_{\tau=1}^t \sum_{i=1}^m b_{i\tau} - \sum_{\tau=1}^t \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau}^G > m\overline{D} \max\left\{2, \sum_{\tau=1}^t |\mathcal{NA}_\tau|\right\} \equiv K_t$$

implies that all pairs in \mathcal{NA}_t from period t can be assigned.

It is now easy to see that each K_t can be bounded from above by a constant independent of n . This, together with Assumption 8.5.2, implies that the necessary capacities are indeed present with probability one if n goes to infinity. \square

In Theorem 5.4.10 we show that the greedy heuristic combined with the sequential procedure are asymptotically optimal with probability one.

Theorem 8.5.4 *If $\mathcal{C} = \emptyset$ and $\mathcal{S} = \emptyset$, the greedy heuristic given in Section 8.2 combined with the sequential procedure are asymptotically optimal with probability one.*

Proof: The proof is similar to the proof of Theorem 8.4.5. \square

8.5.3 Static customers and seasonal demand pattern

In this section we analyze the special case of the acyclic MPSSP where all customers are static and have the same seasonal demand pattern, i.e., $d_{jt} = \sigma_t d_j$ for each $j = 1, \dots, n$ and $t = 1, \dots, T$ where σ_t are nonnegative constants. As we saw in Chapter 7, this special case of the MPSSP can be formulated as a GAP with a Lipschitz objective function. The asymptotic feasibility and optimality follows as in Section 8.4.

As showed in Theorem 7.4.5, Assumption 8.3.1 reads as follows when all facilities are acyclic and all customers are static with the same seasonal demand pattern.

Assumption 8.5.5 *Assume that*

$$\sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right) > \mathcal{E}(D_1).$$

The following lemma is used in the proof of Theorem 8.5.7.

Lemma 8.5.6 *It holds*

$$\sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right) - \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n D_j \mathbf{X}_{ij1}^{\text{LPR}} > 0$$

with probability one when n goes to infinity.

Proof: Note that

$$\begin{aligned} & \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right) - \frac{1}{n} \sum_{i=1}^m \sum_{j=1}^n d_j x_{ij1}^{\text{LPR}} \\ &= \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right) - \frac{1}{n} \sum_{j=1}^n d_j \left(\sum_{i=1}^m x_{ij1}^{\text{LPR}} \right) \\ &= \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right) - \frac{1}{n} \sum_{j=1}^n d_j. \end{aligned}$$

The result follows by using the Law of the Large Numbers and Assumption 8.5.5. \square

Theorem 8.5.7 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, the greedy heuristic given in Section 8.2 is asymptotically feasible with probability one.*

Proof: Recall that this particular MPSSP can be formulated as a GAP with m agents, n tasks, and a Lipschitz objective function. The capacity of each agent is equal to $\min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t b_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right)$ and the requirement of each task is equal to d_j . The result follows now similarly as Theorem 8.4.4 by using Theorem 8.5.1 and Lemma 8.5.6. \square

Theorem 8.5.8 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, the greedy heuristic given in Section 8.2 is asymptotically optimal with probability one.*

Proof: The proof is similar to the proof of Theorem 8.4.5. □

8.6 Numerical illustrations

8.6.1 Introduction

In this section we will test the behaviour of the greedy heuristic proposed in Section 8.2. On one hand, we would like to illustrate the asymptotic results shown in this chapter. On the other hand, we would like to investigate the performance of the greedy heuristic on classes of the MPSSP for which we were not able to derive any asymptotic result.

The feasibility and the objective value of the solution obtained by this greedy heuristic will be improved by two local exchange procedures. They are inspired on the ones proposed for the CCAP in Section 2.3.2. Recall that the first local exchange procedure tries to assign the tasks on which the greedy heuristic failed. The assignment of each of those tasks to agent i is measured by

$$r(i, \ell) = \begin{cases} \sum_{t=1}^T d_{jt} & \text{if } \ell = j \in \mathcal{S} \\ d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \end{cases}$$

and the best agent, say i_ℓ , is defined as the one minimizing $r(i, \ell)$ over the set of agents. We try to assign the tasks to their most desirable agent in decreasing order of $r(i_\ell, \ell)$, either directly when agent i_ℓ has sufficient capacity available, or by a feasible exchange, if one can be found. The second local exchange procedure tries to improve the objective value of the current solution. The possible exchanges of tasks (ℓ, p) are considered in decreasing order of

$$(f(i_\ell, \ell) + f(i_p, p)) - (f(i_\ell, p) + f(i_p, \ell))$$

where i_ℓ and i_p are the agents to which tasks ℓ and p are assigned in the current solution and $f(i, \ell)$ is the pseudo-cost function used by the greedy heuristic. Recall that each dynamic customer defines T tasks of the form (customer, period)-pairs. Those tasks were only allowed to be exchanged with tasks corresponding to the same period.

We have considered different classes of problem instances for the MPSSP. For all of them, we have generated a set of customers and a set of facilities uniformly in the square $[0, 10]^2$. We have generated for each customer a random demand \mathbf{D}_{jt} in period t from the uniform distribution on $[5\alpha_t, 25\alpha_t]$. We have chosen the vector of seasonal factors to be $\alpha = (\frac{1}{2}, \frac{3}{4}, 1, 1, \frac{3}{4}, \frac{1}{2})^\top$. The costs \mathbf{C}_{ijt} are assumed to be proportional to demand and distance, i.e., $\mathbf{C}_{ijt} = \mathbf{D}_{jt} \cdot \text{dist}_{ij}$, where dist_{ij} denotes the Euclidean distance between facility i and customer j . Finally, we have generated inventory holding costs \mathbf{H}_{it} uniformly from $[10, 30]$.

We have fixed the number of facilities at $m = 5$, and the number of periods at $T = 6$. The number of customers ranges from $n = 50$ until $n = 500$ with incremental steps of 50. For each class of problem instances and each size of the problem we have generated 50 problem instances. All the LP-relaxations were solved with CPLEX 6.5 [33].

8.6.2 The cyclic case

In this section we test the performance of the greedy heuristic together with the two local exchange procedures to improve the current solution described above.

We have considered problem instances for

- the purely dynamic case, i.e., $\mathcal{D} = \{1, \dots, n\}$ and $\mathcal{S} = \emptyset$;
- the purely static case, i.e., $\mathcal{S} = \{1, \dots, n\}$ and $\mathcal{D} = \emptyset$;
- and a mixed case, where the probabilities that a customer is static or dynamic are both equal to $\frac{1}{2}$, i.e., $\mathcal{E}(|\mathcal{D}|) = \mathcal{E}(|\mathcal{S}|) = \frac{1}{2}n$.

We have chosen the capacities $b_{it} = \frac{1}{m} \cdot \beta \cdot n$, where

$$\beta = \delta \cdot \frac{15}{T} \cdot \sum_{t=1}^T \sigma_t.$$

To ensure asymptotic feasibility with probability one of the problem instances generated, we need to choose $\delta > 1$. To account for the asymptotic nature of this feasibility guarantee, we have set $\delta = 1.1$ to obtain feasible problem instances for finite n .

Tables 8.1–8.3 summarize the results. In those tables we have used the following notation. Column *I* indicates the size of the problem, i.e., the numbers of customers. Following this we have four groups of columns reporting information about the LP-relaxation (LPR), the greedy heuristic (G), the local exchange procedure to improve feasibility (F), and the local exchange procedure to improve the objective value (O). In this collection of problem instances for the MPSSP the LP-relaxation was always feasible. In block LPR, we have reported the average computation time used to solve the LP-relaxation. In block G, column *st* reports the status of the solution given by the greedy heuristic, more precisely, column *st* is equal to the number of problem instances for which the greedy heuristic could assign all the tasks, i.e., a feasible solution for the MPSSP was found. Column *er* is the average error bound (measured as the percentage on which the greedy heuristic value exceeds the LP-relaxation value). Obviously, this average was calculated only for the problem instances where a feasible solution could be found. Column *t* is the average time employed by the greedy heuristic. Note that we need to solve the LP-relaxation to obtain the pseudo-cost function. We have reported the running time of the greedy heuristic without including the computation time required by the LP-relaxation. If the greedy heuristic could not assign all the tasks, we called the local exchange procedure to improve feasibility. Observe that this procedure was called the number of times that column *G-st* rests to 50. In block F, similar information as the one given for the greedy

heuristic was reported for this procedure. Column *st* is the number of problem instances for which the procedure could find a feasible solution for the MPSSP. Column *er* is the average error bound which was calculated only for the problem instances where a feasible solution could be found. Column *t* is the average required time. For each problem instance where the greedy heuristic together with the local exchange procedure to improve feasibility could find a feasible solution, we called the local exchange procedure to improve the objective value. In block O, we have reported the average error bound and the average computation time. Finally, column *tt* indicates the average total time required by this solution procedure.

We observe that the greedy heuristic finds always a feasible solution when $n \geq 100$. Only for the static case with 50 customers we found one problem instance where the greedy heuristic failed to assign all tasks. The upper bounds on the error are reasonably good. For all types of problem instances this upper bound is below 4.5% when $n \geq 250$. Observe that it is lower than 2% for the static case. Moreover, as expected, improvements are found when n grows. For example, the upper bound on the error is 0.87% for the static case with 500 customers. As for the numerical illustrations for the GAP in Chapter 5, we need to solve the LP-relaxation to obtain the pseudo-cost function for the greedy heuristic. We have reported the running time of the greedy heuristic without including the computation time required by the LP-relaxation. Observe that the time employed by the greedy heuristic is insignificant compared to solving the LP-relaxation. The local exchange procedure to improve feasibility was only called for one problem instance of the static case with 50 customers, and it showed to be successful. The second local exchange to improve the objective value of the solution at hand shows to be effective. We may observe that the upper bound on the error when $n \geq 150$ is below 0.1%. However, this local exchange procedure can be very time consuming specially for large problem instances of the dynamic case.

8.6.3 The acyclic case

Similar results are given in this section for the acyclic case. We have chosen the capacities $b_{it} = \frac{1}{m} \cdot \beta \cdot n$, where

$$\beta = \delta \cdot 15 \cdot \max_{t=1, \dots, T} \left(\frac{1}{t} \sum_{\tau=1}^t \sigma_{\tau} \right).$$

To ensure asymptotic feasibility with probability one, we need to choose $\delta > 1$. As before, we have set $\delta = 1.1$.

The results are presented in Tables 8.4–8.6. Similar behaviour is observed for the greedy heuristic on problem instances of the acyclic MPSSP. We may see that the greedy heuristic is able to find always a feasible solution for the dynamic case. Recall that we were only able to prove asymptotic feasibility of the greedy heuristic combined with the sequential procedure given in Section 8.5.2. It is also remarkable the behaviour of the local exchange procedure to improve the objective value on the acyclic problem instances. We observe that is much less costly in terms of computation time than for the cyclic problem instances. This can be explained by the fact

that after exchanging the assignment of two tasks we need to calculate the objective function, and then, for each agent, we must find the optimal inventory holding costs. When the agent corresponds to a facility with a cyclic inventory pattern, we can show that there exists a period where its optimal inventory level is equal to zero. Therefore, the optimal inventory costs can be found by solving T acyclic problems.

8.6.4 The acyclic and static case

Until now we have generated problem instances of the MPSSP where the facilities are identical. In this section we will investigate this fact on problem instances where all facilities show an acyclic inventory pattern and all customers are static. Recall that we were only able to derive asymptotic results when all customers showed the same seasonal demand pattern.

The variety of the problem instances is based on the type of facility and the type of demand pattern. More precisely, we have considered:

- non-identical facilities and a general demand pattern;
- non-identical facilities and a seasonal demand pattern;
- identical facilities and a general demand pattern;
- identical facilities and a seasonal demand pattern.

We have generated as before the location of the facilities as well as of the customers, and the costs. With respect to the demand, when all customers exhibit the same seasonal demand pattern, we have generated, for each customer, \mathbf{D}_j from the uniform distribution on $[5, 25]$, and then $\mathbf{D}_{jt} = \sigma_t \mathbf{D}_j$. We have chosen the vector of seasonal factors to be $\sigma = \alpha$ where α was defined in Section 8.6.1. For the more general case, we have done as before, i.e., for each customer we have generated a random demand \mathbf{D}_{jt} in period t from the uniform distribution on $[5\alpha_t, 25\alpha_t]$.

We have chosen the capacities $b_{it} = \omega_i \cdot \beta \cdot n$, where

$$\beta = \delta \cdot 15 \cdot \max_{t=1, \dots, T} \left(\frac{1}{t} \sum_{\tau=1}^t \sigma_\tau \right)$$

with $\omega = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})^\top$ when the facilities are identical and, for the more general case, $\omega = (\frac{1}{10}, \frac{1}{10}, \frac{1}{5}, \frac{1}{5}, \frac{2}{5})^\top$. To ensure asymptotic feasibility with probability one for the two classes earlier mentioned, we need to choose $\delta > 1$. As before, we have set $\delta = 1.1$.

The results are presented in Tables 8.7–8.10. From those we cannot derive any relevant difference due to the type of facility. The fact that the customers may have the same seasonal demand pattern seems to influence the results. We may observe that solving the LP-relaxation is computationally less costly. On the other hand, finding a feasible solution appears to be more difficult. We also observe that the upper bounds on the error are better. For a general demand pattern we were not able to prove asymptotic results but we observe that the upper bounds on the error are relatively good being below 2% for $n \geq 300$.

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.11	49	9.37	0.00	1	7.64	0.00	3.05	0.05	0.21
100	0.22	50	7.52	0.00				1.32	0.24	0.50
150	0.38	50	3.97	0.01				0.58	0.52	0.94
200	0.57	50	2.60	0.01				0.41	1.00	1.61
250	0.78	50	1.48	0.01				0.19	1.54	2.37
300	1.02	50	1.85	0.01				0.21	2.26	3.33
350	1.27	50	0.94	0.01				0.11	3.14	4.46
400	1.60	50	1.45	0.01				0.12	4.05	5.70
450	1.93	50	0.81	0.01				0.08	4.99	6.98
500	2.29	50	0.87	0.01				0.07	6.05	8.40

Table 8.1: Greedy heuristic + improvement phase; cyclic and static

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.13	50	15.68	0.00				2.82	0.13	0.28
100	0.29	50	9.69	0.00				0.87	0.53	0.84
150	0.52	50	8.80	0.01				0.69	1.24	1.79
200	0.78	50	6.73	0.01				0.38	2.45	3.26
250	1.03	50	3.49	0.01				0.20	3.69	4.76
300	1.46	50	4.25	0.01				0.19	5.61	7.12
350	1.74	50	2.58	0.02				0.09	7.36	9.16
400	2.25	50	3.36	0.02				0.20	11.05	13.36
450	2.39	50	1.65	0.02				0.07	13.31	15.76
500	3.15	50	2.33	0.02				0.11	15.76	18.98

Table 8.2: Greedy heuristic + improvement phase; cyclic and mixed

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.08	50	15.41	0.01				1.72	0.35	0.45
100	0.20	50	8.71	0.01				0.73	1.46	1.70
150	0.37	50	7.28	0.01				0.57	3.54	3.94
200	0.64	50	6.04	0.01				0.36	6.90	7.58
250	0.85	50	3.51	0.02				0.18	9.89	10.79
300	1.18	50	3.31	0.02				0.18	15.09	16.33
350	1.47	50	2.27	0.02				0.09	18.89	20.42
400	1.74	50	2.79	0.03				0.18	27.23	29.04
450	2.04	50	1.59	0.03				0.06	34.97	37.10
500	2.90	50	2.08	0.04				0.09	66.34	69.94

Table 8.3: Greedy heuristic + improvement phase; cyclic and dynamic

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.10	49	10.38	0.00	1	3.94	0.00	3.19	0.02	0.17
100	0.21	50	7.55	0.00				1.37	0.07	0.31
150	0.36	50	4.88	0.01				0.92	0.17	0.56
200	0.52	50	3.53	0.01				0.55	0.32	0.88
250	0.72	50	2.26	0.01				0.37	0.48	1.24
300	0.95	50	1.65	0.01				0.23	0.72	1.72
350	1.18	50	1.44	0.01				0.19	1.03	2.26
400	1.51	50	1.61	0.02				0.24	1.36	2.93
450	1.76	50	1.08	0.02				0.13	1.70	3.52
500	2.18	50	1.21	0.02				0.14	2.20	4.44

Table 8.4: Greedy heuristic + improvement phase; acyclic and static

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.10	50	16.53	0.01				3.38	0.04	0.17
100	0.23	50	10.12	0.02				1.05	0.17	0.44
150	0.41	50	8.08	0.03				0.86	0.43	0.90
200	0.63	50	6.41	0.07				0.51	0.79	1.52
250	0.83	50	3.89	0.08				0.23	1.20	2.15
300	1.19	50	3.77	0.17				0.24	1.89	3.29
350	1.46	50	2.71	0.19				0.18	2.62	4.32
400	1.81	50	2.82	0.24				0.21	3.37	5.46
450	2.01	50	1.94	0.34				0.09	4.25	6.65
500	2.39	50	2.45	0.28				0.15	5.79	8.51

Table 8.5: Greedy heuristic + improvement phase; acyclic and mixed

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.08	50	15.92	0.02				2.01	0.10	0.22
100	0.18	50	9.35	0.04				0.84	0.48	0.72
150	0.30	50	7.96	0.05				0.82	1.16	1.54
200	0.54	50	5.70	0.18				0.48	2.21	2.95
250	0.74	50	3.24	0.20				0.24	3.25	4.23
300	1.02	50	3.31	0.42				0.23	5.31	6.77
350	1.23	50	2.44	0.50				0.17	6.94	8.70
400	1.43	50	2.74	0.65				0.22	9.07	11.19
450	1.71	50	1.74	0.84				0.10	12.12	14.73
500	2.35	50	2.19	0.65				0.14	37.70	41.34

Table 8.6: Greedy heuristic + improvement phase; acyclic and dynamic

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.10	50	10.38	0.00				3.31	0.01	0.14
100	0.22	50	5.54	0.00				1.17	0.07	0.32
150	0.38	50	4.81	0.00				0.92	0.17	0.58
200	0.55	50	3.23	0.01				0.43	0.32	0.91
250	0.76	50	2.08	0.01				0.33	0.48	1.28
300	0.99	50	1.48	0.01				0.24	0.67	1.71
350	1.25	50	1.12	0.01				0.19	0.95	2.26
400	1.57	50	1.26	0.02				0.18	1.17	2.80
450	1.89	50	0.84	0.02				0.11	1.66	3.61
500	2.33	50	1.36	0.02				0.15	2.21	4.62

Table 8.7: Greedy heuristic + improvement phase; acyclic, non identical facilities, and general demand pattern

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.11	45	6.32	0.00	2	4.04	0.00	1.34	0.01	0.17
100	0.20	47	2.70	0.00	3	1.60	0.01	0.45	0.06	0.30
150	0.33	50	1.87	0.01				0.33	0.13	0.50
200	0.48	50	2.22	0.01				0.28	0.26	0.78
250	0.65	50	1.45	0.01				0.21	0.44	1.14
300	0.85	50	0.95	0.01				0.15	0.65	1.54
350	1.07	50	0.97	0.01				0.10	0.90	2.02
400	1.27	50	0.79	0.01				0.08	1.12	2.44
450	1.56	50	0.86	0.02				0.10	1.56	3.17
500	1.85	50	0.68	0.02				0.08	1.73	3.65

Table 8.8: Greedy heuristic + improvement phase; acyclic, non identical facilities, and seasonal demand pattern

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.10	49	10.38	0.00	1	3.94	0.00	3.19	0.02	0.14
100	0.22	50	7.55	0.00				1.37	0.07	0.31
150	0.37	50	4.88	0.00				0.92	0.17	0.57
200	0.53	50	3.53	0.01				0.55	0.32	0.89
250	0.72	50	2.26	0.01				0.37	0.48	1.25
300	0.95	50	1.65	0.01				0.23	0.72	1.72
350	1.18	50	1.44	0.01				0.19	1.02	2.26
400	1.51	50	1.61	0.02				0.24	1.36	2.93
450	1.78	50	1.08	0.02				0.13	1.73	3.57
500	2.21	50	1.21	0.02				0.14	2.30	4.59

Table 8.9: Greedy heuristic + improvement phase; acyclic, identical facilities, and general demand pattern

I	LPR	G			F			O		tt
	t	st	er	t	st	er	t	er	t	
50	0.09	45	7.58	0.00	1	3.33	0.00	1.56	0.01	0.12
100	0.19	48	2.92	0.00	1	1.81	0.01	0.46	0.06	0.28
150	0.32	50	2.19	0.00				0.34	0.15	0.50
200	0.46	50	2.38	0.01				0.32	0.29	0.79
250	0.61	50	1.62	0.01				0.22	0.46	1.12
300	0.81	50	1.20	0.01				0.16	0.66	1.52
350	1.04	50	1.09	0.01				0.12	0.99	2.09
400	1.26	50	0.78	0.02				0.09	1.25	2.57
450	1.50	50	1.06	0.02				0.10	1.57	3.13
500	1.79	50	0.91	0.02				0.08	1.97	3.83

Table 8.10: Greedy heuristic + improvement phase; acyclic, identical facilities, and seasonal demand pattern

Chapter 9

A Branch and Price algorithm for the MPSSP

9.1 Introduction

The MPSSP is a suitable optimization model to evaluate the layout of a logistics distribution network with respect to costs. Clearly, the most accurate estimation of the total costs of the logistics distribution network by the MPSSP is through its optimal value. Two different formulations have been given for the MPSSP, a Mixed Integer Programming (MIP) problem with assignment and inventory level variables and a CCAP formulation. Nemhauser [99] claims that a good formulation is one of the requirements for the success of an exact solution procedure for an Integer Programming (IP) problem. In this respect, the LP-relaxation of the IP problem should be a good approximation of the convex hull of the feasible integer solutions. Moreover, he emphasizes that the quality of a formulation should not only be measured by the number of variables and constraints. He gives as an example a distribution problem which could not be solved after 100 hours, and a reformulation of it of a considerably larger size was solved in less than 13 minutes, see Barnhart et al. [10]. In this chapter we investigate a Branch and Price algorithm to solve to optimality a set partitioning formulation of the MPSSP.

A Branch and Price approach is a promising technique for solving large-scale real-life problems which can be formulated as IP problems. Basically, it consists of a Branch and Bound algorithm where the LP-relaxation at each node of the tree is solved by a Column Generation procedure. In the following, we will show that issues like branching are not so straightforward as in a standard Branch and Bound scheme. Those two techniques, Branch and Price for IP problems and Column Generation for LP problems, are specially meant for models where the number of variables (columns) is excessively large, but the number of constraints (size of the columns) is relatively small. Most of the applications of the column generation technique have been to set partitioning problems. Gilmore and Gomory [61] were the pioneers in applying a column generation approach to solve the LP-relaxation of an IP problem,

namely, the cutting-stock problem. More recently, it has been used to solve successfully applications, for example, in routing problems (see Desrochers, Desrosiers and Solomon [35]), vehicle and crew scheduling problems (see Freling [51]), and aircrew rostering problems (see Gamache et al. [53]).

Solving an LP problem with a large number of columns to optimality can be computationally very hard. If the number of constraints is relatively small, we know that only few columns (not more than the number of constraints) have a positive value in a basic optimal solution. A column generation approach consists of solving a reduced LP problem where only some of the columns of the original LP problem are present. The optimal solution obtained for the reduced LP problem is a feasible solution for the original LP problem, but it may not be optimal. A so-called pricing problem is defined to search for new columns pricing out, i.e., columns so that by adding them to the reduced LP problem its optimal objective value decreases. If the pricing problem does not find any column pricing out, we have at hand the optimal objective value of the original LP problem. Observe that the pricing problem may need to be called a large number of times. Therefore, the success of the column generation procedure depends critically on the ability to solve the pricing problem efficiently. The main goal of this chapter is to analyze the structure of the pricing problem for the MPSSP.

The dual information of the reduced LP problem is required to define the pricing problem. Therefore, we always must ensure that the reduced LP problem has at least a feasible solution. For problems where this is a critical issue, we can add artificial variables with a high cost (for a minimization formulation) so that they will have a strictly positive value in the optimal solution of the LP problem if and only if this is infeasible. By keeping these artificial variables at each reduced LP problem, we can also ensure that this problem is always feasible. Barnhart et al. [9] suggest adding a column with all components equal to one for the set partitioning formulation.

Another important issue when designing a Branch and Price algorithm is the branching rule. When the optimal solution for the LP-relaxation is fractional we need to branch to find the optimal solution of the IP problem. After branching, new columns may need to be generated in the new nodes of the tree. Standard branching rules may destroy the structure of the pricing problem obtained for the root node. Savelsbergh [121] illustrates this issue for the GAP. Recall that the GAP is a member of the class of CCAP's. He proposes a Branch and Price algorithm for the set partitioning formulation of the GAP. Recall that the columns are defined as possible assignments of tasks to agents. He shows that the pricing problem in the root node turns out to be a Knapsack Problem (KP). Moreover, he argues that branching on the variables of the set partitioning formulation is not compatible with this pricing problem. Consider an agent i , a column ℓ for this agent and the corresponding variable in the set partitioning formulation y_i^ℓ . Suppose that in the optimal solution for the LP-relaxation the variable y_i^ℓ is fractional and is the one selected to branch on. A standard branching rule would create two new nodes where this variable is fixed to 0 and to 1, respectively. He mentions that it is quite likely that the pricing problem again generates the column associated with this variable. Therefore, we need in order to find a column with the second minimum reduced cost, which is not

a KP anymore. Barnhart et al. [9] have unified the literature on Branch and Price algorithms for large scale (Mixed) Integer Problems. One of the central points in their study is finding branching rules compatible with the pricing problem.

In Chapter 2 we have generalized the Branch and Price algorithm for the GAP that was developed by Savelsbergh [121] to a much richer class of problems, the class of convex capacitated assignment problems. The MPSSP, as a member of this class, can be solved to optimality using this procedure. As mentioned above, the viability of this approach depends critically on the possibility of solving the pricing problem efficiently. The purpose of this chapter is to analyze the pricing problem for the MPSSP. We have identified in Section 2.5.4 an important subclass of CCAP's for which this is the case. For the members of this subclass, the pricing problem is equal to a Penalized Knapsack Problem (PKP) which was studied in Section 2.5.5. This is a nonlinear problem which shows nice properties as the KP. The most relevant is that its relaxation can be explicitly solved. Some variants of the MPSSP belong to this subclass. For the rest of MPSSP's the pricing problem is still a generalization of the KP. Based on that we will propose a class of greedy heuristics to find good solutions for the pricing problem.

The outline of this chapter is as follows. In Section 9.2 we show that the pricing problem for the static and seasonal MPSSP can be formulated as a PKP. In Section 9.3 we investigate the structure of the pricing problem for a general MPSSP. We propose a class of greedy heuristics for the pricing problem. A description of the implementation of a Branch and Price algorithm is given in Section 9.4 and some numerical results are given for the acyclic, static, and seasonal MPSSP. Some of the results in this chapter can be found in Freling et al. [52].

9.2 The pricing problem for the static and seasonal MPSSP

Due to the assignment structure in the CCAP, the pricing problem will read, in general, as a generalization of the KP. In this section we will analyze what the pricing problem looks like for the MPSSP when all customers are static and have the same seasonal demand pattern. We will show that the pricing problem can be formulated as a PKP which was studied in Section 2.5.5.

A CCAP is defined by giving a convex cost function g_i and a set of linear constraints for each agent $i = 1, \dots, m$. Recall that the facilities can be seen as the agents in the CCAP formulation of the MPSSP. The static and seasonal MPSSP belongs to the class of CCAP's by choosing the cost function equal to

$$g_i(z) = \sum_{j=1}^n \left(\sum_{t=1}^T c_{ijt} \right) z_j + H_i(z) \quad \text{for each } z \in \mathbb{R}^n$$

and the linear constraints equal to

$$\sum_{j=1}^n d_j z_j \leq \frac{\sum_{\tau=1}^T b_{i\tau}}{\sum_{\tau=1}^T \sigma_\tau} \quad \text{if } i \in \mathcal{C}$$

$$\sum_{j=1}^n d_j z_j \leq \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t b_{i\tau}}{\sum_{\tau=1}^t \sigma_\tau} \right) \quad \text{if } i \notin \mathcal{C}$$

where the function H_i was defined in Section 6.3. (Recall that the function H_i is defined in \mathbb{R}^{nT} . However, when all customers are static, the assignments can be represented by a vector in \mathbb{R}^n . With some abuse of notation, we will still write $H_i(z)$ with $z \in \mathbb{R}^n$ for this case.) Recall that the function H_i calculates the minimal inventory holding costs at facility i to be able to supply the demand of the customers assigned to it.

From Proposition 6.3.4 we know that the function H_i is convex. In fact, it is easy to show that this function is also piecewise linear. This is illustrated by an example for an acyclic facility. We suppress the index i for convenience. Consider $n = 1$, $T = 3$, and

$$\begin{aligned} \sigma &= (1, 1, 1)^\top \\ h &= (1, 1, 1)^\top \\ d_1 &= 25 \\ b &= (50, 20, 10)^\top. \end{aligned}$$

In that case, we have that $H(z_1)$ is equal to the optimal value of

$$\text{minimize } I_1 + I_2 + I_3$$

subject to

$$\begin{aligned} I_1 - I_0 &\leq 50 - 25z_1 \\ I_2 - I_1 &\leq 20 - 25z_1 \\ I_3 - I_2 &\leq 10 - 25z_1 \\ I_0 &= 0 \\ I_t &\geq 0 \quad t = 1, 2, 3. \end{aligned}$$

Figure 9.1 plots the optimal objective function value of its LP-relaxation as a function of the fraction z_1 of the item added to the knapsack. This is a piecewise linear function in z_1 where each breakpoint corresponds to a new inventory variable becoming positive. In this particular case, all inventory variables are equal to zero if the fraction of the demand supplied is below 0.4, i.e., $z_1 \in [0, 0.4]$. If $z_1 \in (0.4, 0.6]$, I_2 becomes positive. Finally, if $z_1 \in (0.6, 1]$, I_1 also becomes positive.

The feasible region of the assignments for facility i is defined by a knapsack constraint while the objective function is defined as the sum of a linear term and a convex function. When all customers are static and have the same demand pattern, it is easy to see that the convex term, H_i , can be indeed written as a convex function of the use of the knapsack $\sum_{j=1}^n d_j z_j$. Therefore, the static and seasonal MPSSP belongs to the subclass of convex capacitated assignment problems introduced in Section 2.5.4 where the pricing problem exhibits a nice structure. We showed that the pricing problem is a PKP. This is a generalization of the KP where a convex

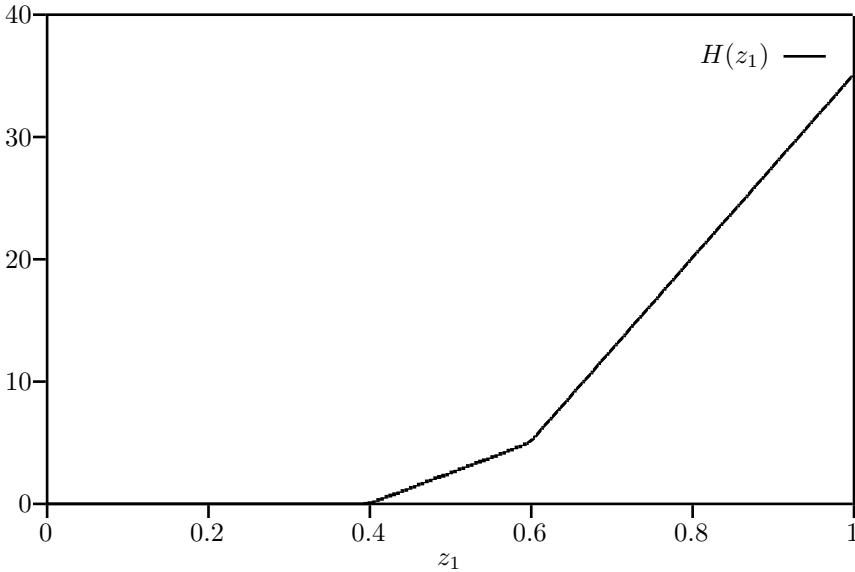


Figure 9.1: The optimal inventory holding costs

penalty for using the knapsack capacity is subtracted from the objective function. To be more precise, in the KP a profit is obtained each time that an item is added to the knapsack. In the PKP, we also have to pay for the capacity used. As for the KP, we have found an explicit expression of the optimal solution of the relaxation of the PKP, see Proposition 2.5.2.

Therefore, we can solve the PKP to optimality by using a Branch and Bound scheme where the relaxations are solved explicitly using the procedure given in Section 2.5.5. Recall that the optimal solution for the relaxation has a similar structure as the one for the LP-relaxation of the KP. We have proved in Proposition 2.5.2 that there exists at most one fractional (critical) item so that only the items whose profit/demand-ratio is not smaller than the critical one are added to the knapsack. The procedure to solve the relaxation of the PKP consists of determining the critical item, say k , and finding then the optimal fraction of this item added to the knapsack, say f^* . In the second step, we have to optimize the function $P^k(\gamma)$ on the segment $[0, 1]$. This function is defined as the objective value of the solution of the PKP where all items with profit/demand-ratio not smaller than the ratio of item k are completely added to the knapsack, as well as a fraction γ of item k . From the discussion above, we know that this function is concave and piecewise linear. Therefore, when optimizing it we only need to evaluate the breakpoints.

In Section 9.3.4 we will also propose a class of greedy heuristics for the pricing problem for the MPSSP based on the one defined by Rinnooy Kan, Stougie and Vercellis [108] for the Multi-Knapsack Problem.

9.3 The pricing problem for the MPSSP

9.3.1 General case

In this section we will examine the structure of the pricing problem for a general multi-period single-sourcing problem. In particular, we will show that the pricing problem is equal to a generalization of the KP for each cyclic facility and of the Multi-Period Binary Knapsack Problem for each acyclic one. Moreover, we will propose a class of greedy heuristics to obtain *good* solutions for the pricing problem.

A pricing problem is associated with some facility i , and it is defined by the cost function and the set of linear constraints corresponding to this facility. The formulation of the pricing problem for the CCAP was given in Section 2.5.2. As we did in Section 2.5.4, we can transform it into an equivalent maximization problem and, without loss of optimality, leave out constant terms. For simplicity, we will still call this problem the pricing problem. The objective function is given by the difference between a linear term and the cost function of facility i . Recall that the cost function of facility i is equal to the summation of a linear term and a convex function. After grouping the linear terms, the objective function of the pricing problem is the difference between a linear term and a convex one. The feasible region of the pricing problem consists of the set of linear constraints associated with the facility and the zero-one constraints. Those linear constraints define the domain of the objective function, i.e., the set of nonnegative vectors so that the objective function is finite. The pricing problem reads as follows:

$$\text{maximize } \sum_{t=1}^T \sum_{j=1}^n p_{jt} z_{jt} - H_i(z)$$

subject to

$$\begin{aligned} z &\in \text{dom}(H_i) \\ z_{jt} &= z_{j1} & j \in \mathcal{S}; t = 2, \dots, T \\ z_{jt} &\in \{0, 1\} & j = 1, \dots, n; t = 1, \dots, T \end{aligned}$$

or equivalently,

$$\text{maximize } \sum_{t=1}^T \sum_{j=1}^n p_{jt} z_{jt} - H_i(z)$$

subject to

(PP)

$$\begin{aligned} \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} &\leq \sum_{t=1}^T b_{it} & i \in \mathcal{C} \\ \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} &\leq \sum_{\tau=1}^t b_{i\tau} & i \notin \mathcal{C}; t = 1, \dots, T \\ z_{jt} &= z_{j1} & j \in \mathcal{S}; t = 2, \dots, T \\ z_{jt} &\in \{0, 1\} & j = 1, \dots, n; t = 1, \dots, T. \end{aligned} \tag{9.1}$$

In the original formulation of the pricing problem exactly one variable is assigned to each task. Recall that each static customer defines exactly one task and each dynamic one defines T tasks. For notational convenience, when both static and dynamic customers are present, we have associated T variables to each task defined by a static customer. Constraints (9.1) ensure that all binary variables corresponding to the same static customer are equal.

We have proved in Proposition 6.3.4 that the function H_i , calculating the optimal inventory holding costs for a given vector of assignments, is convex. From the proof of this result, we can derive that those costs can be indeed expressed as a convex function of the vector of used capacities in each period, say $(\sum_{j=1}^n d_{j1}z_{j1}, \dots, \sum_{j=1}^n d_{jT}z_{jT})$.

The feasible region of the pricing problem differs depending on the type of facility. For a cyclic facility, apart from constraints (9.1) and the Boolean constraints, we have a knapsack constraint. Therefore, the pricing problem is a nonlinear KP where the objective function is equal to the difference between a linear term and a convex function. When the facility is acyclic, we obtain T knapsack constraints instead of just one. We could interpret this as a knapsack with a variable capacity through a planning horizon. Capacity can be transferred to future periods but this has to be paid, see the function H_i .

Faaland [40] and Dudzinski and Walukiewicz [36] have studied similar problems to (PP). The model considered by Dudzinski and Walukiewicz [36] coincides with (PP) when the facility is acyclic, all customers are dynamic, and the function H_i is equal to 0. Faaland [40] just considers integer variables instead of binary ones. They call this problem the Multi-Period (Binary) Knapsack Problem.

9.3.2 The static case

In this section we illustrate with an example what the pricing problem looks like for a problem instance of a cyclic and static MPSSP with a general demand pattern. Recall that the pricing problem is associated with some facility i . For clarity of the exposition, we have suppressed the index i . The number of customers and the planning horizon are fixed to 3. Suppose that after grouping terms, writing the problem on a maximization form and eliminating constants terms the pricing problem reads

$$\text{maximize } 15z_1 + 14z_2 + 16z_3 - H(z)$$

subject to

$$\begin{aligned} 17z_1 + 11z_2 + 16z_3 &\leq 20 \\ z_j &\in \{0, 1\} \quad j = 1, 2, 3 \end{aligned}$$

where $H(z)$, $z = (z_j) \in \mathbb{R}^3$, is the optimal objective value of the linear programming problem

$$\text{minimize } I_1 + 2I_2 + 3I_3$$

subject to

$$I_1 - I_0 \leq 13 - (10z_1 + 5z_2 + 2z_3)$$

$$\begin{aligned}
I_2 - I_1 &\leq 6 - (5z_1 + 3z_2 + 10z_3) \\
I_3 - I_2 &\leq 1 - (2z_1 + 3z_2 + 4z_3) \\
I_0 &= I_3 \\
I_t &\geq 0 \quad t = 1, 2, 3.
\end{aligned}$$

The pricing problem has been reformulated as a nonlinear KP where the items to include in the knapsack are the customers. The objective function is again defined as the difference between a linear term and a convex function. In contrast to the PKP, the convex term representing the optimal inventory holding costs cannot be written as a convex function of the use of the knapsack. As we have mentioned in the previous section, we only can show that it can be written as a convex function of the vector $(\sum_{j=1}^n d_{j1}z_j, \dots, \sum_{j=1}^n d_{jT}z_j)$.

The optimal solution of the relaxation of the pricing problem is equal to $z_1 = 1$, $z_2 = \frac{3}{11}$, and $z_3 = 0$. Recall that the optimal solution of the relaxation of the PKP is so that items are added to the knapsack in decreasing order with respect to the profit/demand-ratio. We add items to the knapsack according to this order if they are feasible (there is enough capacity to add them) and they are profitable (the maximal profit is reached when they are completely added). We stop when we find the first item which does not satisfy these two conditions. We proved that the remaining items should not be added, but just the feasible and profitable fraction of the item where we have stopped. In our example, item 2 has the highest profit/demand-ratio, $\frac{14}{11}$. Suppose that no item has been added to the knapsack yet. Then, there is enough space to add item 2. Moreover, the function $P^2(\gamma)$ calculating the profit of adding fraction γ of the item 2 to the knapsack is equal to

$$P^2(\gamma) = \begin{cases} 14\gamma & \text{if } 0 \leq \gamma \leq \frac{1}{3} \\ 14\gamma - 2(3\gamma - 1) & \text{if } \frac{1}{3} \leq \gamma \leq 1, \end{cases}$$

which is an increasing function in γ . Therefore, if no item was added to the knapsack, the highest profit is obtained by adding item 2 completely. However, in the optimal solution we have only added $\frac{3}{11}$ of the item. Hence, the procedure described above to solve the relaxation of the PKP does not apply to the relaxation of the pricing problem in the cyclic and static MPSSP.

9.3.3 Dynamic case

In this section we give an example of a pricing problem for a problem instance of a cyclic and dynamic MPSSP with a general demand pattern. We consider a similar setting to the static example given in the previous section. The pricing problem is equal to

$$\text{maximize } 2z_{11} + 7z_{21} + 2z_{31} + 10z_{12} + z_{22} + 5z_{32} + 3z_{13} + 6z_{23} + 9z_{33} - H(z)$$

subject to

$$\begin{aligned}
10z_{11} + 5z_{21} + 2z_{31} + 5z_{12} + 3z_{22} + \\
+ 10z_{32} + 2z_{13} + 3z_{23} + 4z_{33} &\leq 20 \\
z_{jt} &\in \{0, 1\} \quad j, t = 1, 2, 3
\end{aligned}$$

where $H(z)$, $z = (z_{jt}) \in \mathbb{R}^{3 \times 3}$, is the optimal objective value of the linear programming problem

$$\text{minimize } I_1 + 2I_2 + 3I_3$$

subject to

$$\begin{aligned} I_1 - I_0 &\leq 13 - (10z_{11} + 5z_{21} + 2z_{31}) \\ I_2 - I_1 &\leq 6 - (5z_{12} + 3z_{22} + 10z_{32}) \\ I_3 - I_2 &\leq 1 - (2z_{13} + 3z_{23} + 4z_{33}) \\ I_0 &= I_3 \\ I_t &\geq 0 \quad t = 1, 2, 3. \end{aligned}$$

Again, the pricing problem has been reformulated as a nonlinear KP where the items to include in the knapsack are the (customer,period)-pairs. The objective function is again defined as the difference between a linear term and a convex function.

The optimal solution of the relaxation of the pricing problem is equal to $z_{11} = \frac{3}{5}$, $z_{21} = 1$, $z_{31} = 1$, $z_{12} = 1$, $z_{32} = \frac{1}{10}$, $z_{33} = \frac{1}{4}$, and the rest of the variables equal to zero. We can derive two conclusions from this example. In contrast to the PKP, the optimal solution of the relaxation of the pricing problem may have more than one split item. Moreover, as for the cyclic and static MPSSP with general demand pattern, the procedure to solve the relaxation of the PKP does not apply to the relaxation of the pricing problem in the cyclic and dynamic MPSSP. In our example, the pair (3,3) is the item with the highest profit/demand-ratio, $\frac{9}{4}$. Suppose that no item has been added to the knapsack yet. Then, there is enough space to add item (3,3). Moreover, the function $P^{(3,3)}(\gamma)$ calculating the profit of adding fraction γ of the item (3,3) to the knapsack is equal to

$$P^{(3,3)}(\gamma) = \begin{cases} 9\gamma & \text{if } 0 \leq \gamma \leq \frac{1}{4} \\ 9\gamma - 2(4\gamma - 1) & \text{if } \frac{1}{4} \leq \gamma \leq 1, \end{cases}$$

which is an increasing function in γ . Therefore, if no item was added to the knapsack, the highest profit is obtained by adding item (3,3) completely. However, in the optimal solution we have only added $\frac{1}{4}$ of this item.

For a given period $t' = 1, \dots, T$, we can show that items are added to the knapsack in decreasing order with respect to the profit/demand-ratio and that there exists just one split item. The proof is straightforward by observing that the subproblem obtained when fixing the solution values of the variables corresponding to periods $t = 1, \dots, T$ and $t \neq t'$ is a PKP.

9.3.4 A class of greedy heuristics

In this section we describe a class of greedy heuristics for (PP) similar to the one proposed by Rinnooy Kan, Stougie and Vercellis [108] for the Multi-Knapsack Problem.

Similarly as in Chapter 2, we will define the concept of feasibility and profitability of an item. Consider the case where some items have been already added to the

knapsack. We will say that an item not yet in the knapsack is a *feasible* item if the capacity constraint(s) are not violated when adding it to the knapsack. Moreover, we will say that it is *profitable* if the maximum profit is obtained when adding it complete to the knapsack.

Let $\mu \in \mathbb{R}_+^T$, and let $\sum_{t=1}^T p_{jt} - \sum_{t=1}^T \mu_t d_{jt}$ if $j \in \mathcal{S}$ and $p_{jt} - \mu_t d_{jt}$ if $j \in \mathcal{D}$ and $t = 1, \dots, T$ be a weight function measuring the value of adding item ℓ to the knapsack, where $\ell = j$ if $j \in \mathcal{S}$ and $\ell = (j, t)$ if $j \in \mathcal{D}$ and $t = 1, \dots, T$. We order the set of items according to non-increasing value of the weight function. Each time an item is added to the knapsack, it could happen that some of the remaining items cannot (or should not) be added anymore because there is not enough capacity, or they are not profitable because the payment for using extra capacity is larger than the benefit of adding them to the knapsack. For all those items j which cannot be added, the variables z_j are forced to 0.

The weight functions introduced above are closely related to the family of pseudo-cost functions defining the class of greedy heuristics for the GAP discussed in Chapter 5 and the one for the MPSSP introduced in Chapter 8. When choosing $\mu_t = 0$ for each $t = 1, \dots, T$ we consider the best item to be added to the knapsack the one maximizing the profit. When $\mu_t = M$ for all $t = 1, \dots, T$, as M grows large, we decide in function of the used capacity. In general, our weight function tries to consider those two measures jointly. Observe that the pricing problem for the MPSSP can be reformulated as a Mixed Integer Problem by including inventory variables. Similarly as for the GAP and the MPSSP, we expect a good behaviour for the greedy heuristic with $\mu = \mu^*$, where $\mu^* \in \mathbb{R}_+^T$ represents the optimal dual subvector corresponding to the capacity constraints in the LP-relaxation of the MIP formulation of the pricing problem.

When the greedy heuristics cannot find columns pricing out and the pricing problem is not equivalent to a PKP, we can still solve it to optimality with a standard optimization package.

9.4 Numerical illustrations

9.4.1 Introduction

In this section we will test a Branch and Price algorithm on problem instances of the static and seasonal MPSSP. We have chosen all facilities to be acyclic, but the scheme also applies when all facilities are cyclic.

The problem instances have been generated according to the stochastic model proposed in Chapter 8 for the MPSSP. We have generated a set of customers and a set of facilities uniformly in the square $[0, 10]^2$. For each customer we have generated the total demand \mathbf{D}_j from the uniform distribution on $[5, 25]$, and then the demand in period t is equal to $\mathbf{D}_{jt} = \sigma_t \mathbf{D}_j$. We have chosen the vector of seasonal factors to be $\sigma = (\frac{1}{2}, \frac{3}{4}, 1, 1, \frac{3}{4}, \frac{1}{2})^\top$. The costs \mathbf{C}_{ijt} are assumed to be proportional to demand and distance, i.e., $\mathbf{C}_{ijt} = \mathbf{D}_{jt} \cdot \text{dist}_{ij}$, where dist_{ij} denotes the Euclidean distance between facility i and customer j . Finally, we have generated inventory holding costs \mathbf{H}_{it} uniformly from $[10, 30]$.

We have chosen the capacities $b_{it} = \frac{1}{m} \cdot \beta \cdot n$, where

$$\beta = \delta \cdot 15 \cdot \max_{t=1, \dots, T} \left(\frac{1}{t} \sum_{\tau=1}^t \sigma_{\tau} \right).$$

To ensure asymptotic feasibility with probability one, we need to choose $\delta > 1$. To account for the asymptotic nature of this feasibility guarantee, we have set $\delta = 1.1$ to obtain feasible problem instances for finite n .

9.4.2 Description of the implementation

We have run the greedy heuristic for the MPSSP described in Chapter 8 to get an initial set of columns for the set partitioning formulation in the root node. Recall that we have shown that this greedy heuristic is asymptotically feasible and optimal with probability one, see Theorems 8.5.7 and 8.5.8. We have already remarked that this greedy heuristic does not guarantee a feasible solution for the assignment constraints. Therefore, the (partial) solution given by the greedy heuristic was improved by the two local exchange procedures described in Section 8.6. The same procedure (the greedy heuristic and the two local exchange procedures) has been applied in each node of the tree with depth at most 10 to improve the best integer solution found by the Branch and Price algorithm.

When, for a given set of columns, we do not have a certificate of optimality of the reduced problem, we search (for each facility) for columns pricing out. The procedure we use for each facility is as follows. We first run a greedy heuristic for the PKP which belongs to the class proposed in Section 9.3.4. Recall that the items should be ordered according to non-increasing value of some weight function. In the current implementation of the Branch and Price we have, for reasons of computational efficiency, simply chosen the weight function equal to p_j . When the obtained column does not price out, we use a Branch and Bound procedure for the PKP with depth-first search. We have branched on the variable equal to one from the optimal solution of the relaxation of the PKP (see Martello and Toth [89]). The relaxation of the PKP was solved explicitly as shown in Section 2.5.5. Without extra computational effort we were able to add more than one column pricing out. More precisely, we have added all columns pricing out from the sequence of improving solutions found in the tree.

With respect to the branching rule, we have chosen the variable x_{ij} which is closest to 0.5. (Recall that when all customers are static we can substitute x_{ijt} by x_{ij} for each $i = 1, \dots, m$, $j = 1, \dots, n$ and $t = 1, \dots, T$.) Preliminary tests have indicated that this is a good choice.

To avoid a large number of columns in the model, we have included two types of deletions of columns. The first one concerns the new columns added to the model in each iteration of the column generation procedure. Each time that the number of columns added to the model is larger than η^1 , we eliminate a fraction v^1 of the ones with reduced costs larger than ζ_1 . The second deletion affects all the columns in the model and works in a similar way. It is applied when the number of columns is larger

than $\eta_1^2, \eta_2^2, \dots$. We have chosen, $\eta^1 = 10, \eta_1^2 = 1000, \eta_2^2 = 2000, \dots, v^1 = v^2 = 0.9$ and $\zeta_1 = \zeta_2 = 0.99$.

All the runs were performed on a PC with a 350 MHz Pentium II processor and 128 MB RAM. All LP-relaxations were solved using CPLEX 6.5 [33]. In contrast with most literature on column generation, we have compared the performance of our Branch and Price algorithm with the performance of the MIP solver from CPLEX applied to the standard formulation of the MPSSP. The objective value of the solution given by the greedy heuristic for the MPSSP was given to CPLEX as an upper bound. Our computational experiences have shown us that both procedures find most of the times the optimal solution in an early stage, however to prove optimality can be very time consuming for some problem instances. Thus, our Branch and Price algorithm and CPLEX as a MIP solver, were stopped when the relative upper bound on the error of the best integer solution found was below 1%.

9.4.3 Illustrations

We have generated 50 random problem instances for each size of the problem. For all of them the number of periods T was fixed to 6. We have generated two classes of problem instances. In the first class we fix the ratio between the number of customers and the number of facilities, and in the second one we fix the number of facilities. Table 9.1 shows results of the performance of our Branch and Price algorithm and CPLEX as a MIP solver for $n/m = 5$, and similarly, Table 9.2 for $n/m = 10$, Table 9.3 for $m = 5$, and Table 9.4 for $m = 10$.

In the tables we have used the following notation. Column I indicates the size of the problem, in the format $m.n$, and column fI indicates the number of these problem instances that are feasible. Next, column $f(h)$ tells us the number of times that the greedy heuristic applied to the MPSSP could find a feasible solution, column $f(r)$ is the number of times that we have a feasible solution in the root node, column f is the number of times that the Branch and Price algorithm could find a feasible solution for the problem, and column s is the number of problem instances that were solved successfully, i.e., either a solution with guaranteed error less than 1% was found, or the problem instance was shown to be infeasible. The following two columns give average results on the quality of the initial solutions: column $er(h)$ is the average upper bound on the error of the initial solution given by the greedy heuristic, and column $er(r)$ gives the upper bound on the error of the solution obtained in the root node. The latter two averages have been calculated only taking into account the problem instances where a feasible solution was found. The following group of columns gives information on the Branch and Price phase of the algorithm. Column $\#c$ is the average number of columns in the model at the end of the Branch and Price procedure, column $\#n$ is the average number of nodes inspected, and column nt shows us how many times the optimal solution of the MPSSP was found in the root node. The final columns pertaining to the Branch and Price algorithm deal with computation times. Column $t(h)$ is the average time used by the greedy heuristic for the MPSSP applied in the root node, and t is the average total time used by the Branch and Price procedure. To illustrate the stability of this average, we have also

calculated the average time of the 45 fastest problem instances, see column t_τ which eliminates the few problem instances that have an extreme effect on the average running time. Finally, the last results show the behaviour of CPLEX as a MIP solver. Column $f(c)$ indicates the number of times that CPLEX could find a feasible solution, and column $s(c)$ shows the number of times that CPLEX was successful (similar to column s above). Column $t(c)$ is the average total time employed by CPLEX and column $t_\tau(c)$ is the average of the 45 fastest problem instances.

The main conclusion that we can draw from Tables 9.1-9.4 is that the Branch and Price algorithm is very well suited for solving this particular variant of the MPSSP, especially when the ratio between the number of customers and the number of facilities is not too large. For large ratios the MIP solver in CPLEX is the more efficient solution approach to this variant of the MPSSP. The breakpoint lies somewhere between the ratios 5 and 10. In fact, CPLEX tends to become more efficient, even in an absolute sense, as the number of customers grows for a fixed number of facilities. A possible explanation for this fact is that CPLEX, as well as the greedy heuristic, seem to be able to take advantage of the fact that, with an increase in the number of customers, the number of feasible options for choosing which sets of customers to assign to a given facility also increases – not only due to the increasing number of customers, but also due to an increased flexibility in switching customers between facilities. On the other hand, for the Branch and Price algorithm this increasing number of feasible assignments translates to an increase in the number of columns in the set partitioning problem, and thus in the number of columns that may need to be generated in the column generation phase.

The second conclusion that can be drawn from the tables is that the Branch and Price algorithm is much more successful in solving the problems than CPLEX. In fact, the Branch and Price algorithm succeeded in finding a solution with an error of at most 1% or giving a certificate of infeasibility of the problem instance for all of the problem instances generated, while CPLEX often failed (due to a lack of memory) to solve the problem satisfactorily – especially for the larger problem instances, with failure rates up to 48% for problem instances with 10 facilities.

Thirdly, the Branch and Price algorithm shows more stability in the computation times, caused by fewer and/or less extreme outliers.

I	B&P										CPLEX			
	fl	f(h)	f(r)	f	s	er(h)	er(r)	#c	#n	nt	t(h)	t	t _r	t _r (c)
2.10	41	39	41	41	50	1.29%	0.00%	44.78	0.74	50	0.01	0.06	0.05	0.05
3.15	41	40	40	41	50	2.46%	0.01%	110.46	0.82	49	0.02	0.11	0.10	0.12
4.20	43	41	43	43	50	3.97%	0.79%	211.36	1.16	44	0.04	0.26	0.23	0.37
5.25	38	37	37	38	50	1.93%	0.61%	237.14	1.26	39	0.10	0.56	0.45	1.00
6.30	46	43	44	46	50	2.45%	1.01%	395.36	3.10	34	0.08	1.37	0.87	6.99
7.35	49	44	44	49	50	2.40%	1.40%	498.12	4.42	24	0.11	2.18	1.60	30.10
8.40	48	48	48	48	50	3.53%	2.56%	591.50	4.44	21	0.13	2.94	2.44	52.09
9.45	49	46	47	49	50	3.67%	3.33%	872.26	23.46	9	0.18	16.23	6.58	478.71

Table 9.1: $n/m = 5$

I	B&P										CPLEX			
	fl	f(h)	f(r)	f	s	er(h)	er(r)	#c	#n	nt	t(h)	t	t _r	t _r (c)
2.20	41	41	41	41	50	0.61%	0.00%	95.70	0.34	50	0.04	0.22	0.14	0.08
3.30	45	44	44	45	50	1.06%	0.13%	457.36	0.98	45	0.05	1.56	1.18	0.27
4.40	48	48	48	48	50	1.41%	0.31%	818.08	1.18	43	0.07	3.77	3.29	0.75
5.50	50	48	48	50	50	1.54%	1.00%	979.56	3.96	30	0.11	13.25	9.25	3.15
6.60	50	48	48	50	50	1.21%	0.93%	998.36	3.00	30	0.16	17.90	14.35	8.74
7.70	49	49	49	49	50	1.45%	1.15%	1274.70	5.32	25	0.23	35.04	28.39	15.99
8.80	50	50	50	50	50	1.86%	1.46%	1708.92	6.30	20	0.30	63.19	54.34	77.29
9.90	50	49	49	50	50	1.40%	1.16%	1924.38	6.60	25	2.15	99.98	74.96	194.94

Table 9.2: $n/m = 10$

B&P														CPLEX			
I	fl	f(h)	f(r)	f	s	er(h)	er(r)	#c	#n	nt	t(h)	t	t _r	f(c)	s(c)	t(c)	t _r (c)
5.25	38	37	37	38	50	1.93%	0.61%	237.14	1.26	39	0.10	0.56	0.45	38	50	8.44	1.00
5.30	41	39	40	41	50	2.35%	0.83%	405.14	1.52	37	0.09	1.15	0.93	41	50	14.76	1.26
5.35	44	42	42	44	50	2.03%	1.09%	588.76	2.92	34	0.08	2.27	1.64	44	50	14.45	1.61
5.40	47	47	47	47	50	1.36%	0.57%	641.22	1.84	41	0.09	3.08	2.32	47	50	3.32	2.07
5.45	48	48	48	48	50	1.08%	0.34%	885.88	0.98	45	0.10	4.46	3.75	48	50	3.33	1.17
5.50	50	49	49	50	50	1.08%	0.53%	954.18	1.52	39	0.11	8.43	7.08	50	50	6.41	2.32
5.55	49	46	46	49	50	1.35%	0.66%	1334.34	4.28	31	0.13	22.25	16.12	49	50	90.93	3.81
5.60	50	50	50	50	50	0.61%	0.38%	1245.00	1.06	42	0.13	16.67	14.32	50	50	4.40	2.39
5.65	49	49	49	49	50	0.92%	0.51%	1308.68	1.26	40	0.15	26.90	21.77	49	50	2.66	2.32

Table 9.3: $m = 5$

		B&P										CPLEX					
I	fl	f(h)	f(r)	f	s	er(h)	er(r)	#c	#n	nt	t(h)	t	t _r	f(c)	s(c)	t(c)	t _r (c)
10.25	38	27	29	38	50	5.91%	1.75%	191.06	2.22	33	0.30	0.91	0.74	37	34	2345.88	1594.68
10.30	43	27	31	43	50	4.33%	1.77%	265.24	4.40	25	0.25	1.73	1.20	39	30	2819.48	1709.18
10.35	48	41	43	48	50	5.99%	4.24%	432.74	9.30	14	0.17	3.59	2.34	46	34	2233.45	1494.59
10.40	44	36	36	44	50	5.70%	3.85%	499.02	11.04	17	0.22	5.29	3.36	41	32	1621.04	1193.54
10.45	49	45	45	49	50	3.85%	3.38%	740.48	22.18	9	0.21	12.73	6.34	48	27	1696.40	1384.96
10.50	48	46	46	48	50	3.17%	2.67%	852.20	13.66	12	0.23	11.00	7.45	48	28	1844.87	1459.34
10.55	46	45	45	46	50	3.34%	3.09%	934.30	20.80	12	0.26	17.74	10.23	46	36	1125.14	842.64
10.60	47	45	45	47	50	2.82%	2.28%	1294.58	32.24	13	0.30	37.52	18.90	46	26	1420.94	1277.64
10.65	50	48	48	50	50	2.66%	2.57%	1201.68	25.96	6	0.30	38.69	25.53	49	34	891.78	743.60

Table 9.4: $m = 10$

Part IV

Extensions

Chapter 10

Additional constraints

10.1 Introduction

Several processes are involved when satisfying the demand of a set of customers. In short, they are the production, the storing, and the delivery processes, although they may be still decomposed into other subprocesses. For example, bottling and canning processes are present when dealing with soft drinks. Those processes imply the utilization of machinery, warehouses, and transportation modes which face some constraints. The multi-period single-sourcing problems introduced in Chapter 6 and analyzed in Chapters 7–9 are optimization models minimizing the total costs involved when performing those processes. For clarity of the exposition, only production capacity constraints were taken into account. In this chapter we will expand the model to be able to deal with additional constraints. In particular, we will study the addition to the original MPSSP proposed in Chapter 6 of throughput and physical capacity constraints at the warehouses. Moreover, we will also study additional constraints to handle products which suffer from perishability due to deterioration or consumer preferences.

The purpose of this chapter is to analyze whether the MPSSP with additional constraints can still be reformulated as a CCAP. In this case, as we did for the original MPSSP, we can apply all the results derived for the CCAP in Chapter 2 to it. These results concern the solution of the CCAP and the generation of suitable problem instances for this problem to test solution procedures. First, we can define a general stochastic model for the MPSSP with additional constraints and find an implicit tight condition to ensure asymptotic feasibility in the probabilistic sense of the problem instances generated by it, see Theorem 2.2.4. The stochastic model is similar to the one proposed for the original MPSSP in Chapter 7; we only need to specify the conditions for the parameters of the additional constraints.

Next, we can define a class of greedy heuristics to obtain feasible solutions for the MPSSP with additional constraints, see Section 2.3.1. Recall that the assignment of a task to an agent was measured by a pseudo-cost function, and the best agent for a task was defined as the one minimizing this pseudo-cost function. The so-called

desirability of assigning a task was defined as the difference between the second smallest and the smallest values of the pseudo-cost function over the set of agents. The tasks were assigned to their most desirable agent in non-increasing order with respect to the desirabilities. The values of the desirabilities must be calculated taking into account that the two most desirable agents for each task should be feasible. Therefore, the additional constraints must also be satisfied by the two most desirable agents. Moreover, the pseudo-cost function plays an important role in the quality of the solution given by the greedy heuristic.

Finally, we can use the Branch and Price scheme developed for the CCAP to solve the MPSSP with additional constraints. This algorithm solves the set partitioning formulation given for the CCAP in Section 2.4 to optimality by a Branch and Bound scheme where the LP-relaxations are solved by a Column Generation procedure. Two critical factors have been analyzed in Chapter 2. The structure of the pricing problem is a major issue in the success of the column generation procedure. Moreover, standard branching rules may destroy the structure of the pricing problem. In this respect, we have shown that a similar branching rule to the one proposed by Savelsbergh [121] for the CCAP formulation of the GAP is compatible with the pricing problem of any CCAP.

There are still some open questions related to the stochastic model, the class of greedy heuristics, and the Branch and Price scheme. As we have mentioned above, the condition to ensure asymptotic feasibility in the probabilistic sense of the problem instances generated by the stochastic model is implicit. It involves minimizing a nonlinear function over a simplex, see Theorem 2.2.4. For several variants of the original MPSSP, we successfully obtained easy-to-check feasibility conditions, see Chapter 7. It is interesting to investigate whether we can learn from the results obtained in Chapter 7 to derive more explicit feasibility conditions for the MPSSP with additional constraints. In particular, we have been able to obtain explicit feasibility conditions for the static and seasonal MPSSP with additional constraints. Nevertheless, more work has to be devoted to other variants of this problem.

A second question of interest is the definition of pseudo-cost functions so that the corresponding greedy heuristics obtain *good* feasible solutions for the MPSSP with additional constraints. In Chapter 8 we have proposed a pseudo-cost function for the original MPSSP so that the corresponding greedy heuristic is asymptotically feasible and optimal in the probabilistic sense for many variants of the original MPSSP. The definition of this pseudo-cost function is based on the information obtained from the LP-relaxation of the original MPSSP. Therefore, the question is whether we can define a pseudo-cost function for the MPSSP with additional constraints so that we can prove a similar result. We have been able to prove asymptotic feasibility and optimality of a greedy heuristic for the static and seasonal MPSSP with additional constraints. Further attention must be dedicated to the other cases.

Finally, one of the main concerns when solving the MPSSP with additional constraints using a Branch and Price scheme is the ability to solve the pricing problem efficiently. In Section 2.5.4, we have identified a subclass of CCAP's for which this is the case. The pricing problem is equivalent to a Penalized Knapsack Problem (PKP). We have shown in Proposition 2.5.2 that the optimal solution of the relaxation of

this problem has a similar structure to the optimal solution of the LP-relaxation of the Knapsack Problem. In Chapter 9 we have shown that some variants of the original MPSSP belong to this subclass of CCAP's. In that chapter, we have also proposed a class of greedy heuristics for the pricing problem of the original MPSSP. It is of interest to investigate the structure of the pricing problem for the MPSSP with additional constraints. We have proved that the pricing problem for the static and seasonal MPSSP with additional constraints can be formulated as a PKP. Further research must be devoted to the investigation of other variants.

The outline of this chapter is as follows. In Section 10.2 we will analyze the addition of throughput capacity constraints at the warehouses to the MPSSP. We will give an equivalent CCAP formulation of this problem which consists of the CCAP formulation of the original MPSSP with the addition of the throughput capacity constraints. In Section 10.3 we will study the MPSSP with physical capacity constraints at the warehouses. We will give an equivalent CCAP formulation which differs from the one of the original MPSSP in both the objective function and the feasible region. Finally, we will analyze in Section 10.4 the addition of perishability constraints. We will show similar results to the MPSSP with physical capacity constraints.

10.2 Throughput capacity constraints

10.2.1 Introduction

In this section we will investigate the addition of throughput capacity constraints at the warehouses to the MPSSP. Due to capacity constraints on handling the products at the warehouses, their maximal throughput is limited. Such constraints can easily be added to the MPSSP. If r_{it} is the throughput capacity at warehouse i in period t , then constraints

$$\sum_{j=1}^n d_{jt} x_{ijt} \leq r_{it} \quad i = 1, \dots, m; t = 1, \dots, T \quad (10.1)$$

force the throughput at warehouse i in period t to be below its upper bound r_{it} . By definition these parameters are nonnegative. Recall that (P) is the linear formulation of the original MPSSP, see Section 6.2. We will refer to problem (P) with the addition of constraints (10.1) as (R).

Let us analyze the new piece of input data. For sake of simplicity, we focus on one warehouse and one period of time so that we can ignore the indices i and t . Constraints (10.1) restrict the throughput in an interval of time. During this period customers are supplied so that new shipments from the plants can be handled. This means that a higher frequency of delivery from the warehouses to customers corresponds to a larger throughput capacity (and thus less restrictive constraints (10.1)), i.e., r is larger. Roughly, the maximal throughput r can be calculated as

$$r = \text{physical dimension of the warehouse} \times \text{frequency of delivery to customers.}$$

10.2.2 Reformulation as a CCAP

A similar reformulation to the one given in Theorem 6.3.5 for the original MPSSP holds when adding throughput capacity constraints. We just need, for each $i = 1, \dots, m$, to substitute the function H_i by \tilde{H}_i where, for $z \in \mathbb{R}_+^{nT}$,

$$\tilde{H}_i(z) = \begin{cases} H_i(z) & \text{if } \sum_{j=1}^n d_{jt} x_{ijt} \leq r_{it} \\ \infty & \text{otherwise.} \end{cases} \quad t = 1, \dots, T$$

Recall that to prove that this formulation is a CCAP we need to show that the function \tilde{H}_i is convex and its domain is defined by a set of linear constraints. These results easily follow from the properties of the function H_i . We will present the two extreme cases of the set \mathcal{C} separately.

Corollary 10.2.1 *If $i \in \mathcal{C}$, the domain of the function \tilde{H}_i is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it}, \sum_{j=1}^n d_{jt} z_{jt} \leq r_{it}, t = 1, \dots, T \right\}.$$

Corollary 10.2.2 *If $i \notin \mathcal{C}$, the domain of the function \tilde{H}_i is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=1}^t b_{i\tau}, t = 1, \dots, T, \right. \\ \left. \sum_{j=1}^n d_{jt} z_{jt} \leq r_{it}, t = 1, \dots, T \right\}.$$

Corollary 10.2.3 *The function \tilde{H}_i is convex and Lipschitz.*

Now we are able to prove that the reformulation mentioned above is a CCAP.

Theorem 10.2.4 *The reformulation of (R) given by*

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{i=1}^m \tilde{H}_i(x_{i..})$$

subject to

(R')

$$\begin{aligned} \sum_{i=1}^m x_{ijt} &= 1 & j = 1, \dots, n; t = 1, \dots, T \\ x_{ijt} &= x_{ij1} & i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \\ x_{ijt} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \\ x_{i..} &\in \text{dom}(\tilde{H}_i) & i = 1, \dots, m \end{aligned}$$

is a convex capacitated assignment problem.

Proof: As in Theorem 6.3.5 for the original MPSSP, we need to prove that the function \tilde{H}_i is convex and its domain is defined by linear constraints. This has been shown in Corollaries 10.2.1-10.2.3. \square

As mentioned in the introduction, all the results derived for the CCAP in Chapter 2 are valid for the MPSSP with throughput capacity constraints.

10.2.3 Generating experimental data

The stochastic model for the CCAP for the particular case of the MPSSP with throughput capacity constraints is similar to the one given for the MPSSP in Chapter 7. For each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \overline{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

As before, let b_{it} depend linearly on n , i.e., $b_{it} = \beta_{it}n$, for positive constants β_{it} . Similarly, we will assume that the right-hand sides of the throughput capacity constraints, r_{it} , depend linearly on n , i.e., $r_{it} = \rho_{it}n$ for positive constants ρ_{it} .

As mentioned in the introduction, Theorem 2.2.4 gives an implicit condition to ensure asymptotic feasibility of the problem instances generated by the stochastic model with probability one. More explicit ones have been found for many variants of the original MPSSP in Chapter 7. In particular, when all customers are static and have the same demand pattern, we have been able to reformulate the original MPSSP as a nonlinear GAP with agent-independent requirements (see Section 8.5.3). This was a key result in finding more explicit feasibility conditions for this variant of the original MPSSP. When adding throughput capacity constraints to the static and seasonal MPSSP, we can still reformulate it as a nonlinear GAP with agent-independent requirements, and thus derive more explicit feasibility conditions. We will analyze the cyclic and the acyclic case separately.

Corollary 10.2.5 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (R') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min \left\{ \frac{\sum_{\tau=1}^T \beta_{i\tau}}{\sum_{\tau=1}^T \sigma_{\tau}}, \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right) \right\} > \mathcal{E}(\mathbf{D}_1)$$

and infeasible with probability one if this inequality is reversed.

Corollary 10.2.6 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (R') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min \left\{ \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right), \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right) \right\} > \mathcal{E}(\mathbf{D}_1)$$

and infeasible with probability one if this inequality is reversed.

An open question is whether we can find explicit feasibility conditions for other variants of the MPSSP with throughput capacity constraints.

10.2.4 A class of greedy heuristics

As for the original MPSSP, the class of greedy heuristics developed for the CCAP can be used to find feasible solutions for the MPSSP with throughput constraints. In Section 2.3.1 we presented the general framework for this class. Two main issues need to be addressed for each CCAP, namely, how to check the feasibility of an assignment and what pseudo-cost function to use to evaluate the assignments. When checking the feasibility of the assignment of task j to agent i , we investigate whether agent i is able to supply the requirement of task j knowing that some tasks were already assigned to this agent. This is equivalent to proving that the vector of assignments including task j and the ones already assigned belongs to the domain of the function \tilde{H}_i .

Similarly as for the original MPSSP, the dual programming problem corresponding to the LP-relaxation of (R) suggests a family of pseudo-cost functions for the greedy heuristics. This family reads as follows:

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T (c_{ijt} + (\lambda_{it} + \xi_{it})d_{jt}) & \text{if } \ell = j \in \mathcal{S} \\ c_{ijt} + (\lambda_{it} + \xi_{it})d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$ and $\xi = (\xi_{it}) \in \mathbb{R}_+^{mT}$. We may expect good results when $\lambda = \lambda^*$ and $\xi = \xi^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of (R), and similarly, ξ^* represents the optimal dual subvector corresponding to the throughput capacity constraints. Observe that both types of capacity constraints have been reformulated as \geq -constraints, so that their dual subvectors are nonnegative. (Clearly, if the LP-relaxation of (R) is infeasible, so is (R). Therefore, the pseudo-cost function is well-defined.)

The main question related to those greedy heuristics is whether we are able to theoretically prove some properties about the quality of the obtained solution. In Chapter 8 we have found a good behaviour of the greedy heuristic using the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of (P). We have shown asymptotic feasibility and optimality for many variants of the original MPSSP when the problem instances are generated by the stochastic model for the CCAP for the particular case of the original MPSSP.

In a similar way as for the original MPSSP, the greedy heuristic with $\lambda = \lambda^*$ and $\xi = \xi^*$ is asymptotically feasible and optimal when all customers are static and have the same demand pattern. As for the original MPSSP, the proof of this result is based on showing that the CCAP formulation is a GAP with a Lipschitz objective function. We will present the two extreme cases of the set \mathcal{C} separately. We have proposed a stochastic model for the parameters defining the feasible region of the MPSSP with throughput capacity constraints in Section 10.2.3. Since we allow for dependencies between costs and requirements parameters, we need to redefine the stochastic model rather than simply add distribution assumptions on the costs parameters. Let the

random vectors $(\mathbf{D}_j, \mathbf{C}_j, \gamma_j)$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $[\underline{D}, \overline{D}]^T \times [\underline{C}, \overline{C}]^{mT} \times \{0, 1\}$ where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, $\mathbf{C}_j = (\mathbf{C}_{ijt})_{i=1, \dots, m; t=1, \dots, T}$, $(\mathbf{D}_j, \mathbf{C}_j)$ are distributed according to an absolutely continuous probability distribution and \underline{D} , \overline{D} , \underline{C} and $\overline{C} \in \mathbb{R}_+$. Furthermore, let b_{it} depend linearly on n , i.e., $b_{it} = \beta_{it}n$, for positive constants $\beta_{it} \in \mathbb{R}_+$. Similarly, we will assume that the right-hand sides of the throughput capacity constraints, r_{it} , depend linearly on n , i.e., $r_{it} = \rho_{it}n$ for positive constants ρ_{it} . We will assume that $\underline{D} > 0$.

From the previous section, we know that the following assumption ensures asymptotic feasibility of the problem instances generated by the stochastic model when all facilities have a cyclic inventory pattern, and all customers are static with the same demand pattern.

Assumption 10.2.7 *Assume that*

$$\min_{i=1}^m \left\{ \frac{\sum_{\tau=1}^T \beta_{i\tau}}{\sum_{\tau=1}^T \sigma_{\tau}}, \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right) \right\} > \mathcal{E}(\mathbf{D}_1).$$

Corollary 10.2.8 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 10.2.7 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: The CCAP formulation of this variant of the MPSSP with throughput capacity constraints is a GAP with a nonlinear objective function. Recall that $\tilde{H}_i = H_i$ when the throughput capacity constraints are imposed. Therefore, the objective function of the CCAP formulation is a Lipschitz function (see Proposition 6.3.4). Now the result follows in a similar way as Theorem 8.5.7. \square

We will make a similar assumption for the acyclic case.

Assumption 10.2.9 *Assume that*

$$\min_{i=1}^m \left\{ \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t \beta_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right), \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right) \right\} > \mathcal{E}(\mathbf{D}_1).$$

Corollary 10.2.10 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 10.2.9 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: Similar to the proof of Corollary 10.2.8. \square

We conjecture that this greedy heuristic is asymptotically feasible and optimal for other variants of the MPSSP with throughput capacity constraints.

10.2.5 A Branch and Price scheme

As discussed in the introduction, the Branch and Price scheme given in Section 2.5 for the CCAP can be used to solve the MPSSP with throughput capacity constraints to optimality. We pointed out that the main issue to be analyzed for each CCAP is the structure of the pricing problem. When solving the MPSSP with throughput capacity constraints with this Branch and Price scheme, the pricing problem is similar to the one for the original MPSSP given in Section 9.3. We just need, for each $i = 1, \dots, m$, to replace the function H_i by the function \tilde{H}_i in the objective function and in the feasible region.

The structure of the problem is a critical factor for the success of the column generation procedure used in each node of the tree generated by the Branch and Price algorithm. As mentioned in the introduction, we have identified a subclass of CCAP's for which the pricing can be solved efficiently. As for the original MPSSP, we can prove that some variants of the MPSSP with throughput constraints belong to this subclass. More precisely, when all customers are static with the same seasonal demand pattern, the pricing problem is equivalent to a PKP. This PKP differs with the one for the original static and seasonal MPSSP in the right-hand side of the knapsack constraint defining the feasible region of the PKP. This constraint is equal to

$$\sum_{j=1}^n d_j z_{j1} \leq \min \left\{ \frac{\sum_{\tau=1}^T b_{i\tau}}{\sum_{\tau=1}^T \sigma_{\tau}}, \min_{t=1, \dots, T} \left(\frac{r_{it}}{\sigma_t} \right) \right\}$$

if $i \in \mathcal{C}$ and

$$\sum_{j=1}^n d_j z_{j1} \leq \min \left\{ \min_{t=1, \dots, T} \left(\frac{\sum_{\tau=1}^t b_{i\tau}}{\sum_{\tau=1}^t \sigma_{\tau}} \right), \min_{t=1, \dots, T} \left(\frac{r_{it}}{\sigma_t} \right) \right\}$$

if $i \notin \mathcal{C}$. Moreover, the function H_i must be replaced by the function \tilde{H}_i .

We have also proposed a class of greedy heuristics for the pricing problem for the original MPSSP, see Section 9.3.4. The structure of this class is general enough to be still applicable when the function H_i is replaced by another one. The key results are how to check feasibility and the weight functions used to evaluate the addition of an item to the knapsack. As for the class of greedy heuristics to solve (R'), checking feasibility is equivalent to showing that a certain vector is in the domain of the function \tilde{H}_i which is easy to do since we just need to check a couple of inequalities, see Corollaries 10.2.1 and 10.2.2. Furthermore, we propose the family of weight functions $\sum_{t=1}^T p_{jt} - \sum_{t=1}^T (\mu_t + \delta_t) d_{jt}$ if $j \in \mathcal{S}$ and $p_{jt} - (\mu_t + \delta_t) d_{jt}$ if $j \in \mathcal{D}$ and $t = 1, \dots, T$ where $\mu \in \mathbb{R}_+^T$ and $\delta \in \mathbb{R}_+^T$.

In the worst case, when the greedy heuristics cannot find columns pricing out and the pricing problem is not equivalent to a PKP, we can still solve it to optimality with a standard optimization package.

10.3 Physical capacity constraints

10.3.1 Introduction

In this section we will analyze the addition of physical capacity constraints at each warehouse to the MPSSP. Let \bar{I}_{it} be the physical capacity at warehouse i in period t , then constraints

$$I_{it} \leq \bar{I}_{it} \quad i = 1, \dots, m; t = 1, \dots, T \quad (10.2)$$

force the physical capacity at warehouse i in period t to be below its upper bound \bar{I}_{it} . By definition those parameters are nonnegative. We will refer to problem (P) with the addition of constraints (10.2) as (\bar{P}) .

10.3.2 The optimal inventory holding costs

To be able to rewrite the MPSSP with physical capacity constraints as a CCAP, we will define for each $i = 1, \dots, m$, the function $\bar{H}_i(z)$, $z \in \mathbb{R}_+^{nT}$, to be the optimal value of the following linear programming problem:

$$\text{minimize } \sum_{t=1}^T h_{it} I_t$$

subject to

$$\begin{aligned} I_t - I_{t-1} &\leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt} & t = 1, \dots, T \\ I_0 &= I_T 1_{\{i \in C\}} \\ I_t &\leq \bar{I}_{it} & t = 1, \dots, T \\ I_t &\geq 0 & t = 1, \dots, T. \end{aligned}$$

The function $\bar{H}_i(z)$ almost coincides with the function $H_i(z)$ defined in Section 6.3. In the feasible region defining the value $\bar{H}_i(z)$, we find the additional upper bounds on the inventory variables. Therefore, the domain of the function $\bar{H}_i(z)$ will be contained in the domain of the function H_i . (As in Chapter 6, we will refer to the domain of the function \bar{H}_i as the set of vectors $z \in \mathbb{R}_+^{nT}$ where the function is well-defined, i.e., $\bar{H}_i(z) < \infty$.) In the following we will investigate the domain of the function \bar{H}_i . For clarity of exposition, we will analyze the cyclic and the acyclic cases separately.

Recall that in the cyclic case we are able to produce in “later” (modulo T) periods for usage in “earlier” (modulo T) periods. Therefore, the required demand should be no more than the total capacity. However, when upper bounds on the inventory levels are imposed, we cannot transfer to future (modulo T) periods more capacity than the maximal inventory levels. Thus, it can be expected that we have to impose that the required demand in $r < T$ consecutive (modulo T) periods is at most equal to the production capacity in this planning horizon plus the maximal inventory level

at the preceding period (modulo T). Below we will show that those conditions are indeed necessary and sufficient for characterizing the domain of the function \bar{H}_i for a cyclic facility. Recall that $[t] = (t+1) \bmod T-1$, i.e., $\alpha_{[t-1]} = \alpha_{t-1}$ for $t = 2, \dots, T$, and $\alpha_{[0]} = \alpha_T$. This definition depends on the planning horizon T . When it is clear from the context, we will suppress this dependence, otherwise we will write $[t]_T$.

Lemma 10.3.1 *If $i \in \mathcal{C}$, the domain of the function \bar{H}_i is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{i[t]} + \bar{I}_{i[\underline{t}-1]}, \right. \\ \underline{t} = 1, \dots, T; r = 1, \dots, T-1, \\ \left. \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it} \right\}. \quad (10.3)$$

Proof: Let $z \in \mathbb{R}_+^{nT}$ be a vector in the domain of the function \bar{H}_i . Then, by aggregating the r consecutive production capacity constraints following the \underline{t} -th one, $\underline{t} = 1, \dots, T$ and $r = 1, \dots, T-1$, we obtain the first type of desired inequalities. The second type follows by simply aggregating the T production capacity constraints, and we can conclude that the domain of \bar{H}_i is a subset of (10.3).

Now consider a vector $z \in \mathbb{R}_+^{nT}$ satisfying the conditions

$$\sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{i[t]} + \bar{I}_{i[\underline{t}-1]} \\ \underline{t} = 1, \dots, T; r = 1, \dots, T-1 \quad (10.4)$$

$$\sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it}. \quad (10.5)$$

Similarly as in Lemma 6.3.1, to prove that vector z belongs to the domain of the function \bar{H}_i , it is enough to show that there exists a vector $y \in \mathbb{R}_+^T$ so that

$$y_t \leq b_{it} \quad t = 1, \dots, T \quad (10.6)$$

$$\sum_{t=\underline{t}+1}^{\underline{t}+r} y_{[t]} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} + \bar{I}_{i[\underline{t}+r]} \\ \underline{t} = 1, \dots, T; r = 1, \dots, T-1 \quad (10.7)$$

$$\sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} y_{[t]} + \bar{I}_{i[\underline{t}]} \\ \underline{t} = 1, \dots, T; r = 1, \dots, T-1 \quad (10.8)$$

$$\sum_{t=1}^T y_t = \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \quad (10.9)$$

since in that case the vector $I = (I_t)$ defined by

$$I_t = \left(\sum_{\tau=1}^t y_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \right) - \min_{s=1, \dots, T} \left(\sum_{\tau=1}^s y_{i\tau} - \sum_{\tau=1}^s \sum_{j=1}^n d_{j\tau} z_{j\tau} \right)$$

for each $t = 1, \dots, T$ and $I_0 = I_T$, belongs to the feasible region of the LP problem defining the value $\bar{H}_i(z)$. It is easy to check that I_t is nonnegative and

$$I_t - I_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt}$$

for each $t = 1, \dots, T$. Moreover, the inventory level I_t is below its upper bound if conditions (10.7) and (10.8) are satisfied for $\underline{t} = 1, \dots, T$ and $r = 1, \dots, T - \underline{t}$. Thus, it remains to prove the existence of such a vector y . We will do it by induction on the planning horizon.

For a planning horizon of length 1, it holds trivially by choosing $\tilde{y}_1 = \sum_{j=1}^n d_{j1} z_{j1}$.

Now, we will assume that if the inequality conditions in (10.4) and (10.5) hold for a planning horizon of length t' , then there exists a nonnegative vector $y \in \mathbb{R}_+^{t'}$ so that conditions (10.6)-(10.9) are also satisfied for a planning horizon of length t' . We will show that the same result holds for a planning horizon of length $(t' + 1)$.

We will distinguish two cases depending on the difference between the demand and the capacity in each period. If $\sum_{j=1}^n d_{jt} z_{jt} \leq b_{it}$ for each $t = 1, \dots, t' + 1$, we can define $\tilde{y}_t = \sum_{j=1}^n d_{jt} z_{jt}$. It is easy to show that \tilde{y} satisfies the desired conditions.

Therefore, we will assume that there exists a period $t_0 = 1, \dots, t' + 1$ so that $\sum_{j=1}^n d_{jt_0} z_{jt_0} > b_{it_0}$. In the following, the expression $[t]$ will be calculated for a planning horizon of length $(t' + 1)$ and thus we will write $[t]_{t'+1}$. We will define a new subproblem where period t_0 is not present, the excess demand in period t_0 , say $\sum_{j=1}^n d_{jt_0} z_{jt_0} - b_{it_0}$, is added to the demand in the previous period, say $[t_0 - 1]_{t'+1}$, and the new upper bound on the inventory level in period $[t_0 - 1]_{t'+1}$ is equal to

$$\min \left\{ \bar{I}_{it_0}, \bar{I}_{i[t_0-1]_{t'+1}} - \left(\sum_{j=1}^n d_{jt_0} z_{jt_0} - b_{it_0} \right) \right\}.$$

For notational simplicity we will call u_t the new required capacities, i.e.,

$$u_t = \begin{cases} \sum_{j=1}^n d_{jt} z_{jt} & \text{if } t \neq [t_0 - 1]_{t'+1}, t_0 \\ \sum_{j=1}^n d_{j[t_0-1]_{t'+1}} z_{j[t_0-1]_{t'+1}} + \sum_{j=1}^n d_{jt_0} z_{jt_0} - b_{it_0} & \text{if } t = [t_0 - 1]_{t'+1} \end{cases}$$

and \bar{I}'_t the new upper bounds in the inventory levels, i.e.,

$$\bar{I}'_t = \begin{cases} \bar{I}_{it} & \text{if } t \neq [t_0 - 1]_{t'+1}, t_0 \\ \min \left\{ \bar{I}_{it_0}, \bar{I}_{i[t_0-1]_{t'+1}} - \left(\sum_{j=1}^n d_{jt_0} z_{jt_0} - b_{it_0} \right) \right\} & \text{if } t = [t_0 - 1]_{t'+1}. \end{cases}$$

(Recall that period t_0 is not present in the new subproblem.) Conditions (10.4) and (10.5) hold for that subproblem. Therefore there exists a vector $\alpha \in \mathbb{R}_+^{t'+1}$ so that

$$\begin{aligned}
 \alpha_t &\leq b_{it} & t = 1, \dots, t' + 1; t \neq t_0 \\
 \sum_{t=\underline{t}+1; [\underline{t}]_{t'+1} \neq t_0}^{\underline{t}+r} \alpha_{[t]_{t'+1}} &\leq \sum_{t=\underline{t}+1; [\underline{t}]_{t'+1} \neq t_0}^{\underline{t}+r} u_{[t]_{t'+1}} + \bar{I}'_{[\underline{t}+r]_{t'+1}} \\
 && \underline{t} = 1, \dots, t' + 1; r = 1, \dots, t'; [\underline{t}]_{t'+1} \neq t_0; [\underline{t} + r]_{t'+1} \neq t_0 \\
 \sum_{t=\underline{t}+1; [\underline{t}]_{t'+1} \neq t_0}^{\underline{t}+r} u_{[t]_{t'+1}} &\leq \sum_{t=\underline{t}+1; [\underline{t}]_{t'+1} \neq t_0}^{\underline{t}+r} \alpha_{[t]_{t'+1}} + \bar{I}'_{[\underline{t}]_{t'+1}} \\
 && \underline{t} = 1, \dots, t' + 1; r = 1, \dots, t'; [\underline{t}]_{t'+1} \neq t_0; [\underline{t} + r]_{t'+1} \neq t_0 \\
 \sum_{t=1; t \neq t_0}^{t'+1} y_t &= \sum_{t=1; t \neq t_0}^{t'+1} u_t.
 \end{aligned}$$

We can define

$$\tilde{y}_t = \begin{cases} b_{it_0} & \text{if } t = t_0 \\ \alpha_t & \text{otherwise.} \end{cases}$$

It is easy to see that $\tilde{y} \in \mathbb{R}^{t'+1}$ satisfies conditions (10.6)-(10.9) for a planning horizon of length $(t' + 1)$. \square

Observe that the domain of the function \bar{H}_i , $i \in \mathcal{C}$, consists of two types of constraints where the second one defines the domain of the function H_i .

Similar conditions to the ones obtained for the cyclic case, where production in earlier can only be consumed in later periods, define the domain of the function \bar{H}_i in the acyclic case. For each $i \notin \mathcal{C}$, we will assume that $\bar{I}_{i0} = 0$.

Lemma 10.3.2 *If $i \notin \mathcal{C}$, the domain of the function \bar{H}_i is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{\tau=\underline{t}}^{\bar{t}} \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=\underline{t}}^{\bar{t}} b_{i\tau} + \bar{I}_{i,\underline{t}-1}, \underline{t} = 1, \dots, T; \bar{t} = \underline{t}, \dots, T \right\}. \quad (10.10)$$

Proof: Let $z \in \mathbb{R}_+^{nT}$ be a vector in the domain of the function \bar{H}_i . Then, by aggregating from the \underline{t} -th capacity constraint until the \bar{t} -th one, $1 \leq \underline{t} \leq \bar{t} \leq T$, the desired inequalities follow, and we can conclude that the domain of \bar{H}_i is a subset of (10.10).

Now consider a vector $z \in \mathbb{R}_+^{nT}$ satisfying the conditions

$$\sum_{\tau=\underline{t}}^{\bar{t}} \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=\underline{t}}^{\bar{t}} b_{i\tau} + \bar{I}_{i,\underline{t}-1} \quad \underline{t} = 1, \dots, T; \bar{t} = \underline{t}, \dots, T. \quad (10.11)$$

Similarly as in Lemma 6.3.2, to prove that vector z belongs to the domain of the \bar{H}_i , it is enough to show that there exists a vector $y \in \mathbb{R}_+^T$ so that

$$y_t \leq b_{it} \quad t = 1, \dots, T \quad (10.12)$$

$$\sum_{\tau=1}^t y_{\tau} \geq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \quad t = 1, \dots, T-1 \quad (10.13)$$

$$\sum_{\tau=1}^t y_{\tau} \leq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} + \bar{I}_{it} \quad t = 1, \dots, T-1 \quad (10.14)$$

$$\sum_{\tau=1}^T y_{\tau} = \sum_{\tau=1}^T \sum_{j=1}^n d_{j\tau} z_{j\tau} \quad (10.15)$$

since in that case the vector $I = (I_t)$ defined by

$$I_t = \sum_{\tau=1}^t y_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau}$$

for each $t = 1, \dots, T$ and $I_0 = 0$, belongs to the feasible region of the LP problem defining the value $\bar{H}_i(z)$. It is easy to check that I_t is nonnegative by condition (10.13), $I_t \leq \bar{I}_{it}$ by condition (10.14), and

$$I_t - I_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt}$$

by condition (10.12) for each $t = 1, \dots, T$. Thus, it remains to prove the existence of such a vector y . We will do it by induction on the planning horizon.

For a planning horizon of length 1, it holds trivially by choosing $\tilde{y}_1 = \sum_{j=1}^n d_{j1} z_{j1}$.

Now, we will assume that if the inequality conditions in (10.11) hold for a planning horizon of length t' , then there exists a nonnegative vector $y \in \mathbb{R}_+^{t'}$ so that conditions (10.12)-(10.15) are also satisfied for a planning horizon of length t' . We will show that the same result holds for a planning horizon of length $(t' + 1)$.

We will distinguish two cases depending on the difference between the demand and the capacity in period $t' + 1$. First consider the case where the demand is no more than the capacity, i.e.,

$$\sum_{j=1}^n d_{j,t'+1} z_{j,t'+1} \leq b_{i,t'+1}.$$

By the induction hypothesis there exists a vector $\alpha \in \mathbb{R}_+^{t'}$ such that

$$\begin{aligned} \alpha_t &\leq b_{it} & t = 1, \dots, t' \\ \sum_{\tau=1}^t \alpha_{\tau} &\geq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} & t = 1, \dots, t' - 1 \\ \sum_{\tau=1}^t \alpha_{\tau} &\leq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} + \bar{I}_{it} & t = 1, \dots, t' - 1 \\ \sum_{\tau=1}^{t'} \alpha_{\tau} &= \sum_{\tau=1}^{t'} \sum_{j=1}^n d_{j\tau} z_{j\tau}. \end{aligned}$$

We will define $\tilde{y}_t = \alpha_t$ for each $t = 1, \dots, t'$ and $\tilde{y}_{t'+1} = \sum_{j=1}^n d_{j,t'+1} z_{j,t'+1}$. It is easy to see that $\tilde{y} \in \mathbb{R}^{t'+1}$ is a nonnegative vector satisfying conditions (10.12)-(10.15) for a planning horizon of length $(t' + 1)$.

Next, we will consider the case where

$$\sum_{j=1}^n d_{j,t'+1} z_{j,t'+1} > b_{i,t'+1}.$$

It suffices to show that the excess demand in period $t' + 1$, i.e.,

$$\sum_{j=1}^n d_{j,t'+1} z_{j,t'+1} - b_{i,t'+1}$$

can be supplied in previous periods. For each \underline{t} and \bar{t} so that $\underline{t} = 1, \dots, t' - 1$ and $\bar{t} = \underline{t}, \dots, t' - 1$, we have that

$$\sum_{\tau=\underline{t}}^{\bar{t}} \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=\underline{t}}^{\bar{t}} b_{i\tau} + \bar{I}_{i,\underline{t}-1}$$

and for each $\underline{t} = 1, \dots, t'$ and $\bar{t} = t'$, we have that

$$\sum_{\tau=\underline{t}}^{t'} \sum_{j=1}^n d_{j\tau} z_{j\tau} + \left(\sum_{j=1}^n d_{j,t'+1} z_{j,t'+1} - b_{i,t'+1} \right) \leq \sum_{\tau=\underline{t}}^{t'} b_{i\tau} + \bar{I}_{i,\underline{t}-1}.$$

Therefore, there exists a vector $\alpha \in \mathbb{R}_+^{t'}$ such that

$$\begin{aligned} \alpha &\leq b_{it} & t = 1, \dots, t' \\ \sum_{\tau=1}^t \alpha_\tau &\geq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} & t = 1, \dots, t' - 1 \\ \sum_{\tau=1}^t \alpha_\tau &\leq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} + \bar{I}_{it} & t = 1, \dots, t' - 1 \\ \sum_{\tau=1}^{t'} \alpha_\tau &= \sum_{\tau=1}^{t'} \sum_{j=1}^n d_{j\tau} z_{j\tau} + \left(\sum_{j=1}^n d_{j,t'+1} z_{j,t'+1} - b_{i,t'+1} \right). \end{aligned}$$

We will define $\tilde{y}_t = \alpha_t$ for each $t = 1, \dots, t'$ and $\tilde{y}_{t'+1} = b_{i,t'+1}$. Now, it is easy to see that $\tilde{y} \in \mathbb{R}^{t'+1}$ is a nonnegative vector satisfying conditions (10.12)-(10.15) for a planning horizon of length $(t' + 1)$. \square

When we fix $\underline{t} = 1$ in the expression of the domain of the function \bar{H}_i , $i \notin \mathcal{C}$, we obtain the inequalities

$$\sum_{\tau=1}^{\bar{t}} \sum_{j=1}^n d_{j\tau} z_{j\tau} \leq \sum_{\tau=1}^{\bar{t}} b_{i\tau} \quad \bar{t} = 1, \dots, T$$

which are exactly the constraints defining the domain of the function H_i .

The following result shows some properties of the function \overline{H}_i .

Proposition 10.3.3 *The function \overline{H}_i is convex and Lipschitz.*

Proof: From Lemmas 10.3.1 and 10.3.2, we have that the domain of the function \overline{H}_i is the intersection of halfspaces in \mathbb{R}^{nT} , and thus a convex set.

In a similar fashion as in Lemma 6.3.3, we can express $\overline{H}_i(z)$ as follows

$$\overline{H}_i(z) = \max_{\omega \in \overline{\Omega}_i} \left\{ \sum_{t=1}^T \omega_t \left(\sum_{j=1}^n d_{jt} z_{jt} - b_{it} \right) - \sum_{t=1}^T \mu_t \bar{I}_{it} \right\}$$

where

$$\begin{aligned} \overline{\Omega}_i = \{ \omega \in \mathbb{R}_+^T : & h_{it} + \omega_t + \mu_t - \omega_{t+1} \geq 0, t = 1, \dots, T-1; \\ & h_{iT} + \omega_T + \mu_T - \omega_1 1_{\{i \in \mathcal{C}\}} \geq 0 \}. \end{aligned}$$

As in Proposition 6.3.4, using this expression of $\overline{H}_i(z)$, we can show that the function \overline{H}_i is convex and Lipschitz. \square

10.3.3 Reformulation as a CCAP

A similar reformulation to the one given in Theorem 6.3.5 for the original MPSSP still holds where the function H_i must be replaced by the function \overline{H}_i .

Theorem 10.3.4 *The reformulation of (\overline{P}) given by*

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{i=1}^m \overline{H}_i(x_{i..})$$

subject to

(\overline{P}')

$$\begin{aligned} \sum_{i=1}^m x_{ijt} &= 1 & j = 1, \dots, n; t = 1, \dots, T \\ x_{ijt} &= x_{ij1} & i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \\ x_{ijt} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \\ x_{i..} &\in \text{dom}(\overline{H}_i) & i = 1, \dots, m \end{aligned}$$

is a convex capacitated assignment problem.

Proof: As in Theorem 6.3.5 for the original MPSSP, we need to show that the function \overline{H}_i is convex and its domain is defined by linear constraints. This has been shown in Proposition 10.3.3 and Lemmas 10.3.1 and 10.3.2. \square

Problems (\bar{P}') and (P') differ in both the objective function and the set of constraints faced by each facility. In the objective function we have substituted the function H_i by \bar{H}_i which is still convex. Moreover, the number of constraints faced by each facility has increased but all of them are still linear.

Since we have found an equivalent CCAP formulation of the MPSSP with physical capacity constraints, we can apply the results derived for that problem.

10.3.4 Generating experimental data

The stochastic model for the MPSSP with physical capacity constraints is similar to the one for the original MPSSP. For each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \bar{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

As before, let b_{it} depend linearly on n , i.e., $b_{it} = \beta_{it}n$, for positive constants β_{it} . Furthermore, we will assume that \bar{I}_{it} depends linearly on n , i.e., $\bar{I}_{it} = \eta_{it}n$ for positive constants η_{it} .

Again, the CCAP formulation of the static and seasonal MPSSP with physical capacity constraints is a nonlinear GAP with agent-independent requirements. Thus, as for the original static and seasonal MPSSP, we will be able to derive feasibility conditions. We will present the two extreme cases of \mathcal{C} separately.

Corollary 10.3.5 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (\bar{P}') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min \left\{ \frac{\sum_{\tau=1}^T \beta_{i\tau}}{\sum_{\tau=1}^T \sigma_{\tau}}, \min_{t=1, \dots, T; r=1, \dots, T-1} \left(\frac{\sum_{\underline{t}=t+1}^{\underline{t}+r} \beta_{i[\underline{t}]} + \eta_{i[\underline{t}-1]}}{\sum_{\underline{t}=t+1}^{\underline{t}+r} \sigma_{[\underline{t}]}} \right) \right\} > \mathcal{E}(\mathbf{D}_1)$$

and infeasible with probability one if this inequality is reversed.

A similar result can be found for the acyclic case.

Corollary 10.3.6 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (\bar{P}') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min_{\underline{t}=1, \dots, T; \bar{t}=\underline{t}, \dots, T} \left(\frac{\sum_{\tau=\underline{t}}^{\bar{t}} \beta_{i\tau} + \eta_{i, \underline{t}-1}}{\sum_{\tau=\underline{t}}^{\bar{t}} \sigma_{\tau}} \right) > \mathcal{E}(\mathbf{D}_1)$$

and infeasible with probability one if this inequality is reversed.

An open question is whether we can find explicit feasibility conditions for other variants of the MPSSP with physical capacity constraints.

10.3.5 A class of greedy heuristics

In this section we will analyze how the class of greedy heuristics for the CCAP applies to the MPSSP with physical capacity constraints. Checking feasibility is equivalent to proving that a certain vector is in the domain of the function \bar{H}_i , which is easy to do since we just need to check a couple of inequalities, see Lemmas 10.3.1 and 10.3.2.

The dual programming problem corresponding the LP-relaxation of (\bar{P}) suggests the following family of pseudo-cost functions for the greedy heuristics

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T (c_{ijt} + \lambda_{it} d_{jt}) & \text{if } \ell = j \in \mathcal{S} \\ c_{ijt} + \lambda_{it} d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$. As for the original MPSSP, we may expect good results when $\lambda = \lambda^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of (\bar{P}) . Again the production capacity constraints have been reformulated as \geq -constraints, so that its dual subvector is nonnegative. (Clearly, if the LP-relaxation of (\bar{P}) is infeasible, so is (\bar{P}) . Therefore, the pseudo-cost function is well-defined.) Observe that the family of pseudo-cost functions is just the same as for the original MPSSP. However, the pseudo-cost functions associated with vector λ^* are not the same because the vector λ^* for (P) and the one for (\bar{P}) are dual optimal subvectors to the production capacity constraints for two different LP-relaxations, namely the one for (P) and the one for (\bar{P}) .

Since the CCAP formulation of the static and seasonal MPSSP with physical capacity constraints is a GAP with a Lipschitz objective function, asymptotic feasibility and optimality of the greedy heuristic with $\lambda = \lambda^*$ follows as for the original static and seasonal MPSSP. Again, since we allow for dependencies between costs and requirements parameters, we need to redefine the stochastic model rather than simply add distribution assumptions on the costs parameters to the stochastic model for the requirement parameters given in Section 10.3.4. We will generate the problem instances according to a similar stochastic model to the one proposed in Section 10.2.4 for the MPSSP with throughput capacity constraints. Note that the throughput capacity parameters are not present in the MPSSP with physical capacity constraints. For the physical capacity parameters, say \bar{I}_{it} , we will assume that depends linearly on n , i.e., $\bar{I}_{it} = \eta_{it}n$ for positive constants η_{it} . The two extreme cases of the set \mathcal{C} will be presented separately.

The following assumption ensures asymptotic feasibility of the problem instances generated by the stochastic model when all facilities have a cyclic inventory pattern, and all customers are static with the same demand pattern.

Assumption 10.3.7 *Assume that*

$$\sum_{i=1}^m \min \left\{ \frac{\sum_{\tau=1}^T \beta_{i\tau}}{\sum_{\tau=1}^T \sigma_{\tau}}, \min_{t=1, \dots, T; r=1, \dots, T-1} \left(\frac{\sum_{t=t+1}^{t+r} \beta_{i[t]} + \eta_{i[t-1]}}{\sum_{t=t+1}^{t+r} \sigma_{[t]}} \right) \right\} > \mathcal{E}(\mathcal{D}_1).$$

Corollary 10.3.8 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 10.3.7 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: Similar to the proof of Corollary 10.2.8. \square

This assumption ensures a similar result for the acyclic case.

Assumption 10.3.9 Assume that

$$\sum_{i=1}^m \min_{\underline{t}=1, \dots, T; \bar{t}=\underline{t}, \dots, T} \left(\frac{\sum_{\tau=\underline{t}}^{\bar{t}} \beta_{i\tau} + \eta_{i,\underline{t}-1}}{\sum_{\tau=\underline{t}}^{\bar{t}} \sigma_{\tau}} \right) > \mathcal{E}(\mathbf{D}_1).$$

Corollary 10.3.10 If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 10.3.9 the greedy heuristic is asymptotically feasible and optimal with probability one.

Proof: Similar to the proof of Corollary 10.2.8. \square

We conjecture that this greedy heuristic is asymptotically feasible and optimal for other variants of the MPSSP with physical capacity constraints.

10.3.6 A Branch and Price scheme

The pricing problem for the MPSSP with physical capacity constraints is similar to the one for the original MPSSP where we must replace the function H_i by the function \bar{H}_i in the objective function and in the feasible region. Again the pricing problem is equivalent to a PKP when all the customers are static and have the same seasonal demand pattern. This PKP differs with the one for the original static and seasonal MPSSP in the right-hand side of the knapsack constraint defining the feasible region of the PKP. This constraint is equal to

$$\sum_{j=1}^n d_j z_{j1} \leq \min \left\{ \frac{\sum_{\tau=1}^T b_{i\tau}}{\sum_{\tau=1}^T \sigma_{\tau}}, \min_{\underline{t}=1, \dots, T; r=1, \dots, T-1} \left(\frac{\sum_{t=\underline{t}+1}^{\underline{t}+r} b_{i[t]} + \bar{I}_{i[\underline{t}-1]}}{\sum_{t=\underline{t}+1}^{\underline{t}+r} \sigma_{[t]}} \right) \right\}$$

if $i \in \mathcal{C}$ and

$$\sum_{j=1}^n d_j z_{j1} \leq \min_{\underline{t}=1, \dots, T; \bar{t}=\underline{t}, \dots, T} \left(\frac{\sum_{\tau=\underline{t}}^{\bar{t}} b_{i\tau} + \bar{I}_{i,\underline{t}-1}}{\sum_{\tau=\underline{t}}^{\bar{t}} \sigma_{\tau}} \right)$$

if $i \notin \mathcal{C}$. Moreover, the function H_i must be replaced by the function \bar{H}_i .

As we mentioned in Section 10.2.5, the class of greedy heuristics for the pricing problem for the original MPSSP given in Section 9.3.4 is also suitable for the pricing problem of the MPSSP with physical capacity constraints. Checking the feasibility of the addition of an item to the knapsack is equivalent to showing that a certain vector is in the domain of the function \bar{H}_i . Recall that the items are added to the knapsack according to non-increasing value of the weight function. We propose the family of weight functions $\sum_{t=1}^T p_{jt} - \sum_{t=1}^T \mu_t d_{jt}$ if $j \in \mathcal{S}$ and $p_{jt} - \mu_t d_{jt}$ if $j \in \mathcal{D}$ and $t = 1, \dots, T$ where $\mu \in \mathbb{R}_+^T$.

Again, we can use a standard optimization package to solve the pricing problem when the greedy heuristics cannot find columns pricing out and the pricing problem is not equivalent to a PKP.

10.4 Perishability constraints

10.4.1 Introduction

Due to deterioration or consumer preferences, products may not be useful after some fixed period of time. In the first case the product exhibits a physical perishability while in the second case they are affected by a marketing perishability. In both cases, the storage duration of the product should be limited. Perishability constraints have mainly been taken into account in inventory control, but they can hardly be found in the literature on distribution models. A notable exception is Myers [96], who presents a model where the maximal demand that can be satisfied for a given set of capacities and under perishability constraints is calculated.

The original MPSSP is suitable for products which are not affected by long storage periods. However, modifications must be included when we are dealing with perishable products. When the product has a limited shelf-life, we need to be sure that the time the product is stored is not larger than its shelf-life. When the shelf-life is shorter than a period, we can model the perishability by imposing a lower bound on the throughput. If the shelf-life of the product is equal to k periods, the constraints

$$\sum_{\tau=t+1}^{t+k} \sum_{j=1}^n 1_{\{i \in \mathcal{C} \vee \tau \leq T\}} d_{j[\tau]} x_{ij[\tau]} \geq I_{it} \quad t = 1, \dots, T \quad (10.16)$$

impose that the inventory at warehouse i at the end of period t is at most equal to the total demand supplied out of this warehouse during the k consecutive periods following period t , for each $i = 1, \dots, m$, $t = 1, \dots, T$. (Observe that in the cyclic case, period 1 is the next period to period T .) Hereafter, we will assume that the shelf-life of the product is equal to k periods, $k = 1, \dots, T - 2$. We will refer to problem (P) with the addition of constraints (10.16) as (L).

10.4.2 The optimal inventory holding costs

We will define for each $i = 1, \dots, m$, the function $H_i^k(z)$, $z \in \mathbb{R}_+^{nT}$, to be the optimal value of the following linear programming problem:

$$\text{minimize } \sum_{t=1}^T h_{it} I_t$$

subject to

$$\begin{aligned} I_t - I_{t-1} &\leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt} & t = 1, \dots, T \\ I_t &\leq \sum_{\tau=t+1}^{t+k} \sum_{j=1}^n 1_{\{i \in \mathcal{C} \vee \tau \leq T\}} d_{j[\tau]} z_{j[\tau]} & t = 1, \dots, T \\ I_0 &= I_T 1_{\{i \in \mathcal{C}\}} \\ I_t &\geq 0 & t = 1, \dots, T. \end{aligned}$$

As in the function \overline{H}_i , we have introduced upper bounds on the inventory level variables. In the following, we will investigate the expression of the domain of the function H_i^k for both the cyclic and the acyclic case.

In the following lemma we give an explicit expression of the domain of the function H_i^k for a cyclic facility. As for the function H_i , the required demand should be at most the total capacity. Now we must ensure in addition that quantities shipped to future (modulo T) periods must be consumed in the following (modulo T) k periods. Therefore, the feasibility conditions say that the requirements in $r - k$ ($r < T$) consecutive (modulo T) periods is not more than the production capacity in those periods plus the preceding k (modulo T) periods.

Lemma 10.4.1 *If $i \in \mathcal{C}$, the domain of the function H_i^k is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{i[t]} \right. \quad (10.17)$$

$$\left. \begin{aligned} & \underline{t} = 1, \dots, T; \ r = k+1, \dots, T-1 \\ & \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it} \end{aligned} \right\}.$$

Proof: Let $z \in \mathbb{R}_+^{nT}$ be a vector in the domain of the function H_i^k . Then, there exists a vector $I' \in \mathbb{R}_+^T$ so that

$$I'_t - I'_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt} \quad (10.18)$$

$$I'_t \leq \sum_{\tau=\underline{t}+1}^{\underline{t}+k} \sum_{j=1}^n 1_{\{i \in \mathcal{C} \vee \tau \leq T\}} d_{j[\tau]} z_{j[\tau]} \quad (10.19)$$

for each $t = 1, \dots, T$. Then, by aggregating the r consecutive constraints of the type (10.18) following the \underline{t} -th one, $\underline{t} = 1, \dots, T$ and $r = k+1, \dots, T-1$, we have that

$$\sum_{\tau=\underline{t}+1}^{\underline{t}+r} I'_{[\tau]} - \sum_{\tau=\underline{t}+1}^{\underline{t}+r} I'_{[\tau-1]} \leq \sum_{\tau=\underline{t}+1}^{\underline{t}+r} b_{i[\tau]} - \sum_{\tau=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[\tau]} z_{j[\tau]}$$

which is equivalent to

$$I'_{[\underline{t}+r]} - I'_{[\underline{t}]} \leq \sum_{\tau=\underline{t}+1}^{\underline{t}+r} b_{i[\tau]} - \sum_{\tau=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[\tau]} z_{j[\tau]}$$

and this implies that

$$-I'_{\underline{t}} \leq \sum_{\tau=\underline{t}+1}^{\underline{t}+r} b_{i[\tau]} - \sum_{\tau=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[\tau]} z_{j[\tau]}.$$

Using the upper bound for $I'_{\underline{t}}$ given by condition (10.19) when $t = \underline{t}$, we have that

$$- \sum_{\tau=\underline{t}+1}^{\underline{t}+k} \sum_{j=1}^n d_{j[\tau]} z_{j[\tau]} \leq \sum_{\tau=\underline{t}+1}^{\underline{t}+r} b_{i[\tau]} - \sum_{\tau=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[\tau]} z_{j[\tau]}.$$

Since $r \geq k + 1$, we have derived the first type of desired inequalities. The second type follows by aggregating the T constraints in (10.18), and we can conclude that the domain of H_i^k is a subset of (10.17).

Now consider a $z \in \mathbb{R}_+^{nT}$ such that

$$\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{i[t]} \quad \underline{t} = 1, \dots, T; \quad r = k + 1, \dots, T - 1 \quad (10.20)$$

$$\sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^T b_{it}. \quad (10.21)$$

Similarly as in Lemma 6.3.1, to prove that vector z belongs to the domain of the function H_i^k , it is enough to show that there exists a vector $y \in \mathbb{R}_+^T$ so that

$$\begin{aligned} y_t &\leq b_{it} & t = 1, \dots, T \\ \sum_{t=\underline{t}+1}^{\underline{t}+r} y_{[t]} &\leq \sum_{t=\underline{t}+1}^{\underline{t}+k} \sum_{j=1}^n d_{j[t]} z_{j[t]} + \bar{I}_{i[\underline{t}+r]} & \underline{t} = 1, \dots, T; \quad r = 1, \dots, T - 1 \\ \sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} z_{j[t]} &\leq \sum_{t=\underline{t}+1}^{\underline{t}+r} y_{[t]} + \bar{I}_{i[\underline{t}]} & \underline{t} = 1, \dots, T; \quad r = k + 1, \dots, T - 1 \\ \sum_{t=1}^T y_t &= \sum_{t=1}^T \sum_{j=1}^n d_{jt} z_{jt}, \end{aligned}$$

since in that case the vector $I = (I_t)$ defined by

$$I_t = \left(\sum_{\tau=1}^t y_{i\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} z_{j\tau} \right) - \min_{s=1, \dots, T} \left(\sum_{\tau=1}^s y_{i\tau} - \sum_{\tau=1}^s \sum_{j=1}^n d_{j\tau} z_{j\tau} \right)$$

$t = 1, \dots, T$ and $I_0 = I_T$, belongs to the feasible region of the LP problem defining the value $H_i^k(z)$, and the result follows. The proof is similar to Lemma 10.3.1.

In a similar way as in the proof of Lemma 10.3.1 we can show that if conditions (10.20) and (10.21) hold, then such a vector y exists. \square

In the following lemma we give an expression of the domain of the function H_i^k for an acyclic facility. The conditions are similar to the cyclic case where again we have lost the opportunity to produce in future periods for usage in earlier ones.

Lemma 10.4.2 *If $i \notin \mathcal{C}$, the domain of the function H_i^k is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{it} \right. \\ \left. \begin{aligned} &\underline{t} = 1, \dots, T; r = k+1, \dots, T-\underline{t} \\ &\sum_{t=1}^{\underline{t}} \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^{\underline{t}} b_{it} \quad \underline{t} = 1, \dots, T \end{aligned} \right\}. \quad (10.22)$$

Proof: Let $z \in \mathbb{R}_+^{nT}$ be a vector in the domain of the function H_i^k . Then, there exists a vector $I' \in \mathbb{R}_+^T$ so that

$$I'_t - I'_{t-1} \leq b_{it} - \sum_{j=1}^n d_{jt} z_{jt} \quad (10.23)$$

$$I'_t \leq \sum_{\tau=t+1}^{t+k} \sum_{j=1}^n 1_{\{i \in \mathcal{C} \vee \tau \leq T\}} d_{j\tau} z_{j\tau} \quad (10.24)$$

for each $t = 1, \dots, T$. Let $\underline{t} = 1, \dots, T$, $r \geq k+1$ and $\underline{t}+r \leq T$. Then, by aggregating the r consecutive constraints of the type (10.23) following the \underline{t} -th one we obtain the first type of desired inequalities. The second type follows by simply aggregating the first \underline{t} constraints in (10.23), and we can conclude that the domain of H_i^k is a subset of (10.22).

Now consider a vector $z \in \mathbb{R}_+^{nT}$ satisfying the conditions

$$\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{it} \\ \underline{t} = 1, \dots, T; r = k+1, \dots, T-\underline{t} \quad (10.25)$$

$$\sum_{t=1}^{\underline{t}} \sum_{j=1}^n d_{jt} z_{jt} \leq \sum_{t=1}^{\underline{t}} b_{it} \\ \underline{t} = 1, \dots, T. \quad (10.26)$$

In a similar way as in the proof of Lemma 10.3.2 we can show that if conditions (10.25) and (10.26) hold, then z belongs to the domain of the function H_i^k . \square

It is straightforward to check that the domain of the function H_i^k is contained in the domain of the function H_i for each $i = 1, \dots, m$.

The following result shows some properties of the function H_i^k .

Proposition 10.4.3 *The function H_i^k is convex and Lipschitz.*

Proof: From Lemmas 10.4.1 and 10.4.2, we have that the domain of the function H_i^k is the intersection of halfspaces in \mathbb{R}^{nT} , and then a convex set.

In a similar fashion as in Lemma 6.3.3, we can express $H_i^k(z)$ as follows

$$H_i^k(z) = \max_{\omega \in \Omega_i^k} \left\{ \sum_{t=1}^T \left(\omega_t - \sum_{\ell=1}^k 1_{\{i \in \mathcal{C} \vee \ell < t\}} \alpha_{[t-\ell]} \right) \sum_{j=1}^n d_{jt} z_{jt} - \sum_{t=1}^T \omega_t b_{it} \right\}$$

where

$$\begin{aligned} \Omega_i^k = \{ \omega \in \mathbb{R}_+^T : & h_{it} + \omega_t + \alpha_t - \omega_{t+1} \geq 0, \quad t = 1, \dots, T-1; \\ & h_{iT} + \omega_T + \alpha_T - \omega_1 1_{\{i \in \mathcal{C}\}} \geq 0 \}. \end{aligned}$$

As in Proposition 6.3.4, using this expression of $H_i^k(z)$, we can show that the function H_i^k is convex and Lipschitz. \square

10.4.3 Reformulation as a CCAP

Theorem 10.4.4 gives an equivalent CCAP formulation of the MPSSP with perishability constraints.

Theorem 10.4.4 *The reformulation of (L) given by*

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n c_{ijt} x_{ijt} + \sum_{i=1}^m H_i^k(x_{i..})$$

subject to

(L')

$$\begin{aligned} \sum_{i=1}^m x_{ijt} &= 1 & j = 1, \dots, n; \quad t = 1, \dots, T \\ x_{ijt} &= x_{ij1} & i = 1, \dots, m; \quad j \in \mathcal{S}; \quad t = 2, \dots, T \\ x_{ijt} &\in \{0, 1\} & i = 1, \dots, m; \quad j = 1, \dots, n; \quad t = 1, \dots, T \\ x_{i..} &\in \text{dom}(H_i^k) & i = 1, \dots, m \end{aligned}$$

is a convex capacitated assignment problem.

Proof: The proof is similar to the one of Theorem 10.2.4. \square

As when adding physical capacity constraints, problems (L') and (P') differ in both the objective function and the set of constraints associated with each facility. In the objective function we have substituted the function H_i by the convex function H_i^k . Moreover, each facility faces additional constraints which are still linear.

This formulation of the MPSSP with perishability constraints allows us to make use of the results derived for the CCAP.

10.4.4 Generating experimental data

The stochastic model for the MPSSP with perishability constraints is equal to the one proposed for the original MPSSP, i.e., for each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \overline{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

As before, let b_{it} depend linearly on n , i.e., $b_{it} = \beta_{it}n$, for positive constants β_{it} .

As for the original MPSSP, the fact that the CCAP formulation of the static and seasonal MPSSP with perishability constraints is a nonlinear GAP with agent-independent requirements allows us to derive explicit feasibility conditions for it. We will analyze the two extreme cases of \mathcal{C} separately.

Corollary 10.4.5 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (L') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min \left\{ \min_{\underline{t}=1, \dots, T; \, r=k+1, \dots, T-1} \left(\frac{\sum_{t=\underline{t}+1}^{\underline{t}+r} \beta_{i[t]}}{\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sigma[t]} \right), \frac{\sum_{t=1}^T \beta_{it}}{\sum_{t=1}^T \sigma_t} \right\} > \mathcal{E}(\mathbf{D}_1)$$

and infeasible with probability one if this inequality is reversed.

A similar result can be found for the acyclic case.

Corollary 10.4.6 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, (L') is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{i=1}^m \min \left\{ \min_{\underline{t}=1, \dots, T; \, r=k+1, \dots, T-\underline{t}} \left(\frac{\sum_{t=\underline{t}+1}^{\underline{t}+r} \beta_{it}}{\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sigma_t} \right), \min_{\underline{t}=1, \dots, T} \left(\frac{\sum_{t=1}^{\underline{t}} \beta_{it}}{\sum_{t=1}^{\underline{t}} \sigma_t} \right) \right\} > \mathcal{E}(\mathbf{D}_1)$$

and infeasible with probability one if this inequality is reversed.

An open question is whether we can find explicit feasibility conditions for other variants of the MPSSP with perishability constraints.

10.4.5 A class of greedy heuristics

The class of greedy heuristics for the CCAP can be used for the MPSSP with perishability constraints. Checking feasibility is equivalent to proving that certain vector is in the domain of the function H_i^k , which is easy to do since we just need to check a couple of inequalities, see Lemmas 10.4.1 and 10.4.2.

The dual programming problem corresponding the LP-relaxation of (L) suggests the following family of pseudo-cost functions for the greedy heuristics

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T \left(c_{ijt} + \left(\lambda_{it} - \sum_{\ell=1}^k 1_{\{i \in \mathcal{C} \vee \ell < t\}} \alpha_{i[t-\ell]} \right) d_{jt} \right) & \text{if } \ell = j \in \mathcal{S} \\ c_{ijt} + \left(\lambda_{it} - \sum_{\ell=1}^k 1_{\{i \in \mathcal{C} \vee \ell < t\}} \alpha_{i[t-\ell]} \right) d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$ and $\alpha = (\alpha_{it}) \in \mathbb{R}_+^{mT}$. We may expect good results when $\lambda = \lambda^*$ and $\alpha = \alpha^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of (L), and similarly, α^* represents the optimal dual subvector corresponding to constraints (10.16). Again, both types of capacity constraints have been reformulated as \geq -constraints, so that their dual subvectors are nonnegative. (Clearly, if the LP-relaxation of (L) is infeasible, so is (L). Therefore, the pseudo-cost function is well-defined.)

The greedy heuristic with $\lambda = \lambda^*$ and $\alpha = \alpha^*$ is asymptotically feasible and optimal when all customers are static and have the same demand pattern. Recall that we allow for dependencies between costs and requirements parameters, therefore we need to redefine the stochastic model rather than simply add distribution assumptions on the costs parameters to the stochastic model for the requirement parameters given in Section 10.4.4. We will generate the problem instances according to a similar stochastic model to the one proposed in Section 10.2.4 for the MPSSP with throughput capacity constraints. Note that the throughput capacity parameters are not present in the MPSSP with perishability constraints. The two extreme cases of the set \mathcal{C} will be presented separately.

The following assumption ensures asymptotic feasibility of the problem instances generated by the stochastic model when all facilities have a cyclic inventory pattern, and all customers are static with the same demand pattern.

Assumption 10.4.7 *Assume that*

$$\sum_{i=1}^m \min \left\{ \min_{t=1, \dots, T; r=k+1, \dots, T-1} \left(\frac{\sum_{t=\underline{t}+1}^{t+r} \beta_{i[t]}}{\sum_{t=\underline{t}+k+1}^{t+r} \sigma_{[t]}} \right), \frac{\sum_{t=1}^T \beta_{it}}{\sum_{t=1}^T \sigma_t} \right\} > \mathcal{E}(\mathbf{D}_1)$$

Corollary 10.4.8 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 10.4.7 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: Similar to the proof of Corollary 10.2.8. □

This assumption ensures a similar result for the acyclic case.

Assumption 10.4.9 Assume that

$$\sum_{i=1}^m \min \left\{ \min_{\underline{t}=1, \dots, T; r=k+1, \dots, T-\underline{t}} \left(\frac{\sum_{t=\underline{t}+1}^{\underline{t}+r} \beta_{it}}{\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sigma_t} \right), \min_{\underline{t}=1, \dots, T} \left(\frac{\sum_{t=1}^{\underline{t}} \beta_{it}}{\sum_{t=1}^{\underline{t}} \sigma_t} \right) \right\} > \mathcal{E}(\mathbf{D}_1)$$

Corollary 10.4.10 If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 10.4.9 the greedy heuristic is asymptotically feasible and optimal with probability one.

Proof: Similar to the proof of Corollary 10.2.8. \square

We conjecture that this greedy heuristic is asymptotically feasible and optimal for other variants of the MPSSP with perishability constraints.

10.4.6 A Branch and Price scheme

The pricing problem for the MPSSP with perishability constraints is obtained by replacing in the objective function and in the feasible region of the pricing problem of original MPSSP the function H_i by the function H_i^k . Again the pricing problem is equivalent to a PKP when all the customers are static and have the same seasonal demand pattern. This PKP differs with the one for the original static and seasonal MPSSP in the right-hand side of the knapsack constraint. This constraint is equal to

$$\sum_{j=1}^n d_j z_{j1} \leq \sum_{i=1}^m \min \left\{ \min_{\underline{t}=1, \dots, T; r=k+1, \dots, T-1} \left(\frac{\sum_{t=\underline{t}+1}^{\underline{t}+r} b_{i[t]}}{\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sigma_{[t]}} \right), \frac{\sum_{t=1}^T b_{it}}{\sum_{t=1}^T \sigma_t} \right\}$$

if $i \in \mathcal{C}$ and

$$\sum_{j=1}^n d_j z_{j1} \leq \sum_{i=1}^m \min \left\{ \min_{\underline{t}=1, \dots, T; r=k+1, \dots, T-\underline{t}} \left(\frac{\sum_{t=\underline{t}+1}^{\underline{t}+r} b_{it}}{\sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sigma_t} \right), \min_{\underline{t}=1, \dots, T} \left(\frac{\sum_{t=1}^{\underline{t}} b_{it}}{\sum_{t=1}^{\underline{t}} \sigma_t} \right) \right\}$$

if $i \notin \mathcal{C}$. Moreover, the function H_i must be replaced by the function H_i^k .

Again as mentioned in Section 10.2.5, feasible solutions for the pricing problem for the MPSSP with perishability constraints can be found by the class of greedy heuristics given in Section 9.3.4. Once more, when checking the feasibility of the addition of an item to the knapsack is equivalent to proving that a certain vector is

in the domain of the function H_i^k . With respect to the weight function, we propose the family

$$\begin{cases} \sum_{t=1}^T p_{jt} - \sum_{t=1}^T (\mu_t - \sum_{\ell=1}^k 1_{\{i \in \mathcal{C} \vee \ell < t\}} \alpha_{[t-\ell]}) d_{jt} \\ \quad \text{if } \ell = j \in \mathcal{S} \\ p_{jt} + (\mu_t - \sum_{\ell=1}^k 1_{\{i \in \mathcal{C} \vee \ell < t\}} \alpha_{[t-\ell]}) d_{jt} \\ \quad \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\mu \in \mathbb{R}_+^T$ and $\alpha \in \mathbb{R}_+^T$.

When the greedy heuristics cannot find columns pricing out and the pricing problem is not equivalent to a PKP, we can still solve it to optimality with a standard optimization package.

Chapter 11

A three-level logistics distribution network

11.1 Introduction

The multi-period single-sourcing problems introduced in Chapter 6 and the extensions analyzed in Chapter 10 evaluate the total costs of a logistics distribution network when satisfying the demand of a set of customers. We have assumed in those models that production and storage take place at the same location. Due to this assumption, the problem reduces to the evaluation of the design of a two-level logistics distribution network. We have shown that the MPSSP, as well as the mentioned extensions of it, can be formulated as a CCAP. In this chapter we will study three-level logistics distribution networks in which the plants and the warehouses have been decoupled, see Figure 11.1. When reformulating these models as optimization problems in the assignment variables only, the objective function loses its separability in the warehouses, in contrast to the models analyzed until now. Therefore, the three-level MPSSP and similar extensions to the ones proposed in Chapter 10 cannot be reformulated as a CCAP. Among the results derived for the CCAP in Chapter 2, this separability is only crucial for the Branch and Price scheme since it allows us to give a set partitioning formulation of the problem. Thus, most of the results still apply to models evaluating three-level logistics distribution networks.

The particular scenario we consider in this chapter is the following. A single product is produced by a set of plants which face capacity constraints. These quantities are immediately sent to the warehouses where they can be stored until a customer demand occurs. We do not allow for storage at the plants, transportation between the plants, or transportation between the warehouses. Customers are supplied by the warehouses under single-sourcing conditions. Initially, we will assume that the throughput and the physical capacity constraints are not present in the model. In addition, we will assume that the production costs are linear in the quantity produced, and the transportation costs between the plants and the warehouses are linear in the quantity transported. While the former assumption is a bit restrictive the latter is

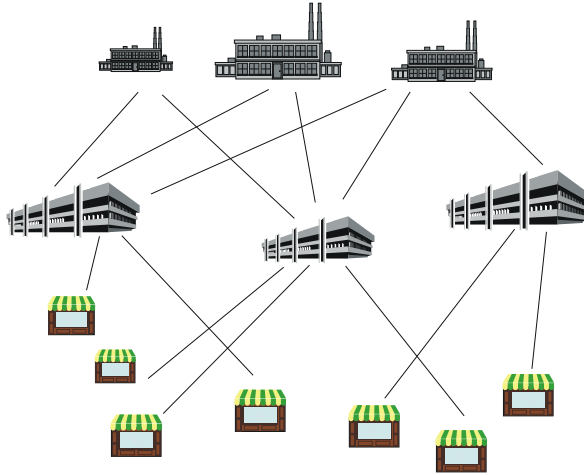


Figure 11.1: Allocations in the logistics distribution network

quite reasonable since the transportation on that level is usually of the full-truckload type. As for the MPSSP, the natural formulation of the three-level MPSSP is mixed integer linear in the production quantities, the inventory levels, and the assignment variables. Similarly as for the MPSSP in Section 6.3.1, there exists an equivalent minimization formulation where the only decision variables are the assignments of the customers to the warehouses. Both the feasible region of the MPSSP and the one of the three-level version are just defined by the assignments constraints, i.e., the ones ensuring that each customer is exactly assigned to one warehouse. The objective function of the reformulation of the three-level MPSSP is again the sum of a linear term and a nonlinear function. For each assignment, the nonlinear term is the optimal solution value of a Multi-Period Transportation Problem and represents the minimal total costs (i.e., production, transportation, and inventory holding costs) needed to supply the quantities implied by this vector to the warehouses. As mentioned above, in contrast to the MPSSP, this nonlinear term is, in general, not separable in the warehouses. Similarly as for the MPSSP, the feasibility of the problem instances of the three-level MPSSP is equivalent to showing that the objective function is well-defined in at least one vector in the feasible region. In contrast to the MPSSP, this can be checked in linear time, in a similar way as for the Transportation Problem (hereafter TP). The LP-relaxation of this model has an interesting property. We will prove that, by fixing the feasible assignments of the optimal solution for the LP-relaxation and arbitrarily assigning the split ones, we obtain a feasible solution for the three-level MPSSP. The relative error in the value of this solution converges to zero as the number of customers goes to infinity. Since the feasible region of the reformulation of the three-level MPSSP is of the type of a CCAP, more feasible solutions can be found by the class of greedy heuristics for the CCAP given in Section 2.3.1. To remark the similarity with the MPSSP, we will prove that a member of this class is feasible and asymptotically optimal for the problem instances generated

by a stochastic model similar to the one proposed for the MPSSP in Section 8.3.

As in Chapter 10, we will enrich the model by adding several types of constraints. As for the MPSSP, we can reformulate the three-level MPSSP with additional constraints similarly to the three-level MPSSP. The reformulation consists of the one given for the three-level MPSSP where the nonlinear term in the objective function must be replaced by another function, reflecting the fact that fewer assignments are feasible due to the added constraints. As in the basic model, this function is not well-defined for the vectors of assignments for which the demand implied cannot be supplied due to the constraints. When the throughput capacity at the warehouses is constrained, it is easy to characterize the set of assignments for which the objective function is well-defined. By adding the throughput capacity constraints to the reformulation of the three-level MPSSP, we obtain an equivalent assignment formulation for the three-level MPSSP with throughput capacity constraints. We will make use of the stochastic model and the class of greedy heuristics for the CCAP. The physical capacity and the perishability constraints seem to be, in principle, more difficult to handle. To find the set of assignment vectors where the objective function is well-defined we need to give an explicit expression of the domain of the nonlinear term in the objective function. In all the two-level formulations we were able to characterize this domain by a set of linear constraints. Based on the feasibility conditions obtained in Chapter 10 for the MPSSP with additional constraints, we will pose some conjectures for the corresponding domains for the three-level MPSSP. We have just been able to prove its correctness for a particular case of the three-level MPSSP with physical capacity constraints. Still, the class of greedy heuristics for the CCAP can be applied to those two models.

The outline of this chapter is as follows. In Section 11.2 we will formulate the three-level MPSSP as a mixed-integer linear programming problem and show the relationship with the CCAP through a reformulation of the model as an assignment problem. We will describe an asymptotically optimal solution method based on altering the optimal solution of the LP-relaxation. We will also analyze how the stochastic model and the class of greedy heuristics for the CCAP apply to the three-level MPSSP. In Section 11.3 we will get some insight about adding throughput capacity constraints at the warehouses. In Section 11.4 we will discuss the physical capacity constraints. Similarly, we will analyze in Section 11.5 the addition of perishability constraints. In Section 11.6 we will present some numerical illustrations of the behaviour of a greedy heuristic for a variant of the three-level MPSSP with throughput capacity constraints. Some of the results in this chapter can be found in Romeijn and Romero Morales [116].

11.2 The three-level MPSSP

11.2.1 The model

Similarly as for (P_0) in Chapter 6, let n denote the number of customers, m the number of warehouses, q the number of plants, and T the planning horizon. The demand of customer j in period t is denoted by d_{jt} , while the production capacity at

plant l in period t is equal to b_{lt} . The unit production, handling, and transportation costs at plant l for transportation to warehouse i in period t are equal to p_{lit} . The costs of assigning customer j to warehouse i in period t are equal to a_{ijt} . As before, the assignment costs can be arbitrary functions of demand and distance. The unit inventory holding costs at warehouse i in period t are equal to h_{it} . (Note that all parameters are required to be nonnegative.)

Recall that $\mathcal{C} \subseteq \{1, \dots, m\}$ is the set of warehouses at which the inventory pattern is restricted to be cyclic and $\mathcal{S} \subseteq \{1, \dots, n\}$ is the set of customers that needs to be assigned to the same warehouse in all periods. Recall that the only interesting and realistic cases for the set \mathcal{C} are the two extremes $\mathcal{C} = \emptyset$ and $\mathcal{C} = \{1, \dots, m\}$. Recall that each customer $j \in \mathcal{S}$ is called static and each customer $j \notin \mathcal{S}$ dynamic.

The three-level MPSSP can now be formulated as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} y_{lit} + \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it} \\ \text{subject to} \quad & \end{aligned} \tag{E}$$

$$\begin{aligned} \sum_{j=1}^n d_{jt} x_{ijt} + I_{it} &= \sum_{l=1}^q y_{lit} + I_{i,t-1} \\ i &= 1, \dots, m; t = 1, \dots, T \end{aligned} \tag{11.1}$$

$$\sum_{i=1}^m y_{lit} \leq b_{lt} \quad l = 1, \dots, q; t = 1, \dots, T \tag{11.2}$$

$$\sum_{i=1}^m x_{ijt} = 1 \quad j = 1, \dots, n; t = 1, \dots, T \tag{11.3}$$

$$x_{ijt} = x_{ij1} \quad i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \tag{11.4}$$

$$\begin{aligned} I_{i0} &= I_{iT} 1_{\{i \in \mathcal{C}\}} \\ i &= 1, \dots, m \end{aligned} \tag{11.5}$$

$$\begin{aligned} x_{ijt} &\in \{0, 1\} \\ i &= 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \end{aligned} \tag{11.6}$$

$$y_{lit} \geq 0 \quad l = 1, \dots, q; i = 1, \dots, m; t = 1, \dots, T$$

$$I_{it} \geq 0 \quad i = 1, \dots, m; t = 1, \dots, T$$

where y_{lit} denotes the quantity produced at plant l and delivered to warehouse i in period t , x_{ijt} is equal to 1 if customer j is assigned to warehouse i in period t and 0 otherwise, and I_{it} denotes the inventory level at warehouse i at the end of period t . Constraints (11.1) model the balance between the inflow, the storage and the outflow at warehouse i in period t . The production quantity at plant l in period t is restricted by (11.2). Constraints (11.3) and (11.6) ensure that each customer is assigned to exactly one warehouse in each period. Moreover, constraints (11.4) force that each static customer is assigned to the same warehouse through the whole planning horizon. For each warehouse $i \in \mathcal{C}$, constraints (11.5) impose that the

inventory levels at the beginning and at the end of the planning horizon are equal, and for each warehouse $i \notin \mathcal{C}$ that the inventory at the beginning of the planning horizon is equal to zero. Since all the parameters are nonnegative, this problem is always well-defined, i.e., its optimal value is bounded from below. Moreover, without loss of optimality we can relax the equations in (11.1) to \leq -constraints.

11.2.2 An equivalent assignment formulation

As for the MPSSP, the optimization program (E) can be reformulated by replacing the production quantities and the inventory levels by nonlinear expressions in the assignment variables. We can easily see that the following is an equivalent formulation for (E):

$$\text{minimize } \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt} + H(x)$$

subject to

$$\begin{aligned} \sum_{i=1}^m x_{ijt} &= 1 & j = 1, \dots, n; t = 1, \dots, T \\ x_{ijt} &= x_{ij1} & i = 1, \dots, m; j \in \mathcal{S}; t = 2, \dots, T \\ x_{ijt} &\in \{0, 1\} & i = 1, \dots, m; j = 1, \dots, n; t = 1, \dots, T \\ x &\in \text{dom}(H) \end{aligned}$$

where the function $H(x)$, $x \in \mathbb{R}_+^{mnT}$, is defined as the optimal value to the following linear programming problem:

$$\text{minimize } \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} y_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it}$$

subject to

$$I_{i,t-1} - I_{it} + \sum_{l=1}^q y_{lit} = \sum_{j=1}^n d_{jt} x_{ijt} \quad i = 1, \dots, m; t = 1, \dots, T \quad (11.7)$$

$$\sum_{i=1}^m y_{lit} \leq b_{lt} \quad l = 1, \dots, q; t = 1, \dots, T \quad (11.8)$$

$$\begin{aligned} I_{i0} &= I_{iT} 1_{\{i \in \mathcal{C}\}} & i = 1, \dots, m \\ y_{lit} &\geq 0 & l = 1, \dots, q; i = 1, \dots, m; t = 1, \dots, T \\ I_{it} &\geq 0 & i = 1, \dots, m; t = 1, \dots, T. \end{aligned}$$

Given a feasible set of assignments $x \in \mathbb{R}^{mnT}$, if $H(x) < +\infty$, then the plants and the warehouses are able to supply the demand pattern implied by the vector of assignments x . Moreover, $\sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt} + H(x)$ is equal to the minimal costs incurred when supplying that demand.

As we did for the MPSSP, we would like to characterize the set of assignment vectors for which the objective function is well-defined, i.e., we would like to characterize the domain of the function H . We will analyze the two extreme cases of the set \mathcal{C} separately.

Lemma 11.2.1 *If $\mathcal{C} = \{1, \dots, m\}$, the domain of the function H is equal to*

$$\left\{ x \in \mathbb{R}_+^{mnT} : \sum_{\tau=1}^T \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{\tau=1}^T \sum_{l=1}^q b_{l\tau} \right\}.$$

Proof: Let x be a vector in the domain of the function H . Then, there exists a vector $(y', I') \in \mathbb{R}_+^{qmT} \times \mathbb{R}_+^{mT}$ so that

$$\begin{aligned} I'_{i,t-1} - I'_{it} + \sum_{l=1}^q y'_{lit} &= \sum_{j=1}^n d_{jt} x_{ijt} \quad i = 1, \dots, m; t = 1, \dots, T \\ \sum_{i=1}^m y'_{lit} &\leq b_{lt} \quad l = 1, \dots, q; t = 1, \dots, T \end{aligned}$$

where $I'_{i0} = I'_{iT}$ for each $i = 1, \dots, m$. The desired inequality follows by aggregating the first type of constraints for each $i = 1, \dots, m$ and $t = 1, \dots, T$ and then using the upper bound on $\sum_{i=1}^m y'_{lit}$ for each $i = 1, \dots, m$.

Suppose now that we have a vector $x \in \mathbb{R}_+^{mnT}$ satisfying the condition

$$\sum_{\tau=1}^T \sum_{i=1}^m \sum_{j=1}^n d_{jt} x_{ij\tau} \leq \sum_{\tau=1}^T \sum_{l=1}^q b_{l\tau}. \quad (11.9)$$

If there exists a vector $y \in \mathbb{R}_+^{qmT}$ so that

$$\sum_{i=1}^m y_{lit} \leq b_{lt} \quad l = 1, \dots, q; t = 1, \dots, T \quad (11.10)$$

and

$$\sum_{\tau=1}^T \sum_{l=1}^q y_{li\tau} = \sum_{\tau=1}^T \sum_{j=1}^n d_{j\tau} x_{ij\tau} \quad i = 1, \dots, m, \quad (11.11)$$

then the vector (y, I) , where

$$I_{it} = \sum_{\tau=1}^t \sum_{l=1}^q y_{li\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{ij\tau} - \min_{s=1, \dots, T} \left(\sum_{\tau=1}^s \sum_{l=1}^q y_{li\tau} - \sum_{\tau=1}^s \sum_{j=1}^n d_{j\tau} x_{ij\tau} \right)$$

for each $i = 1, \dots, m$ and $t = 1, \dots, T$, belongs to the feasible region of the LP problem defining the value $H(x)$ and the result follows. First observe that by definition

I_{it} are nonnegative. The capacity constraints (11.8) are satisfied by using conditions (11.10). We only need to show that constraints (11.7) are also satisfied. For $t = 1$ we have that

$$\begin{aligned}
 I_{i0} - I_{i1} + \sum_{l=1}^q y_{li1} &= \sum_{\tau=2}^T \sum_{l=1}^q y_{li\tau} - \sum_{\tau=2}^T \sum_{j=1}^n d_{j\tau} x_{ij\tau} + \sum_{l=1}^q y_{li1} \\
 &= \sum_{\tau=1}^T \sum_{l=1}^q y_{li\tau} - \sum_{\tau=2}^T \sum_{j=1}^n d_{j\tau} x_{ij\tau} \\
 &= \sum_{\tau=1}^T \sum_{j=1}^n d_{j\tau} x_{ij\tau} - \sum_{\tau=2}^T \sum_{j=1}^n d_{j\tau} x_{ij\tau} \\
 &= \sum_{j=1}^n d_{j1} x_{ij1}.
 \end{aligned}$$

Now, for $t = 2, \dots, T$, the result follows just by simple substitution. Thus, it remains to prove the existence of such vector y . We will do it by induction on the planning horizon.

For a planning horizon of length 1, the inequality conditions in (11.10) and in (11.11) together with the nonnegativity assumption on y define the feasible region of a TP with q sources and m demand points. Moreover, the inequality condition in (11.9) for a planning horizon of length 1 says that the aggregate demand cannot exceed the aggregate capacity, which clearly is a sufficient condition for feasibility of the TP.

Now, we will assume that if condition (11.9) holds for a planning horizon of length t' , then there exists a nonnegative vector $y \in \mathbb{R}_+^{gmt'}$ so that the inequality conditions in (11.10) and in (11.11) are satisfied for $t = 1, \dots, t'$. We will show that the same result holds for a planning horizon of length $t' + 1$.

We will distinguish two cases depending on the difference between the demand and the capacity in each period. First consider the case where the demand in period t is no more than the capacity in the same period, for each $t = 1, \dots, t' + 1$. Then, the result follows trivially by solving a TP for each period.

Suppose now that there exists some $t_0 = 1, \dots, t' + 1$ so that

$$\sum_{i=1}^m \sum_{j=1}^n d_{jt_0} x_{ijt_0} > \sum_{l=1}^q b_{lt_0}.$$

It suffices to show that the excess demand in period t_0 , i.e.,

$$\sum_{i=1}^m \sum_{j=1}^n d_{jt_0} x_{ijt_0} - \sum_{l=1}^q b_{lt_0}$$

can be supplied in the rest of the periods. This is easy to see since

$$\sum_{\tau=1; \tau \neq t_0}^{t'+1} \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} + \left(\sum_{i=1}^m \sum_{j=1}^n d_{jt_0} x_{ijt_0} - \sum_{l=1}^q b_{lt_0} \right) \leq \sum_{\tau=1; \tau \neq t_0}^{t'+1} \sum_{l=1}^q b_{l\tau},$$

and the result follows. \square

A similar result to Lemma 11.2.1 is given for the acyclic case.

Lemma 11.2.2 *If $\mathcal{C} = \emptyset$, the domain of the function H is equal to*

$$\left\{ x \in \mathbb{R}_+^{mnT} : \sum_{\tau=1}^t \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{\tau=1}^t \sum_{l=1}^q b_{l\tau}, t = 1, \dots, T \right\}.$$

Proof: In a similar way as in Lemma 11.2.1 we can show that each vector x in the domain of the function H satisfies the conditions

$$\sum_{\tau=1}^t \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{\tau=1}^t \sum_{l=1}^q b_{l\tau} \quad t = 1, \dots, T. \quad (11.12)$$

Now suppose that we have a vector x which satisfies the inequality conditions in (11.12).

If there exists a vector $y \in \mathbb{R}_+^{qmT}$ so that

$$\sum_{i=1}^m y_{lit} \leq b_{lt} \quad l = 1, \dots, q; t = 1, \dots, T \quad (11.13)$$

and

$$\sum_{\tau=1}^t \sum_{l=1}^q y_{li\tau} \geq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{ij\tau} \quad i = 1, \dots, m; t = 1, \dots, T, \quad (11.14)$$

then the vector (y, I) , where

$$I_{it} = \sum_{\tau=1}^t \sum_{l=1}^q y_{li\tau} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{ij\tau}$$

for each $i = 1, \dots, m$ and $t = 1, \dots, T$, belongs to the feasible region of the LP problem defining the value $H(x)$ and the result follows. By using conditions (11.14) we can derive that I_{it} are nonnegative. The capacity constraints (11.8) are satisfied by using conditions (11.13). We only need to prove that constraints (11.7) are also satisfied. For $t = 1$ we have that

$$\begin{aligned} I_{i0} - I_{i1} + \sum_{l=1}^q y_{li1} &= - \sum_{l=1}^q y_{li1} + \sum_{j=1}^n d_{j1} x_{ij1} + \sum_{l=1}^q y_{li1} \\ &= \sum_{j=1}^n d_{j1} x_{ij1}. \end{aligned}$$

Now, for $t = 2, \dots, T$, the result follows just by simple substitution. Thus, it remains to prove the existence of such vector y . We will do it by induction on the planning horizon.

For a planning horizon of length 1, the existence follows easily as in Lemma 11.2.1.

Now, we will assume that if the inequality conditions in (11.12) hold for a planning horizon of length t' , then there exists a nonnegative vector $y \in \mathbb{R}_+^{qmt'}$ so that the inequality conditions in (11.13) and (11.14) are satisfied for $t = 1, \dots, t'$. We will show that the same result holds for a planning horizon of length $t' + 1$.

We will distinguish two cases, depending on the difference between the aggregate demand and the aggregate capacity in period $t' + 1$. First consider the case where the aggregate demand is no more than the aggregate capacity, i.e.,

$$\sum_{i=1}^m \sum_{j=1}^n d_{j,t'+1} x_{ij,t'+1} \leq \sum_{l=1}^q b_{l,t'+1}.$$

Then there exists a vector $z \in \mathbb{R}_+^{qm}$ such that

$$\sum_{i=1}^m z_{li,t'+1} \leq b_{l,t'+1} \quad l = 1, \dots, q$$

and

$$\sum_{l=1}^q z_{li,t'+1} \geq \sum_{j=1}^n d_{j,t'+1} x_{ij,t'+1} \quad i = 1, \dots, m.$$

Moreover, by the induction hypothesis there exists a vector $y \in \mathbb{R}_+^{qmt'}$ such that

$$\sum_{i=1}^m y_{lit} \leq b_{lt} \quad l = 1, \dots, q; t = 1, \dots, t'$$

and

$$\sum_{\tau=1}^t \sum_{l=1}^q y_{li\tau} \geq \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} x_{ij\tau} \quad i = 1, \dots, m; t = 1, \dots, t'.$$

It is easy to see that $(y, z) \in \mathbb{R}_+^{qm(t'+1)}$ is a nonnegative vector satisfying the inequality conditions in (11.13) and (11.14) for $t = 1, \dots, t' + 1$.

Next, we will consider the case where

$$\sum_{i=1}^m \sum_{j=1}^n d_{j,t'+1} x_{ij,t'+1} > \sum_{l=1}^q b_{l,t'+1}.$$

It suffices to show that the excess demand in period $t' + 1$ can be supplied in previous periods. This is easy to see since

$$\sum_{\tau=1}^t \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{\tau=1}^t \sum_{l=1}^q b_{l\tau}$$

for each $t = 1, \dots, t' - 1$ and

$$\sum_{\tau=1}^{t'} \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} + \left(\sum_{i=1}^m \sum_{j=1}^n d_{j,t'+1} x_{ij,t'+1} - \sum_{l=1}^q b_{l,t'+1} \right) \leq \sum_{\tau=1}^{t'} \sum_{l=1}^q b_{l\tau}$$

for $t = t'$. \square

The following result shows that the three-level MPSSP is feasible if and only if the total demand is below the total capacity when $\mathcal{C} = \{1, \dots, m\}$. Similarly, if $\mathcal{C} = \emptyset$ the necessary and sufficient conditions for feasibility read as the aggregate demand until period t below the aggregate capacity until period t , for each $t = 1, \dots, T$.

Proposition 11.2.3 *If $\mathcal{C} = \{1, \dots, m\}$, the three-level MPSSP is feasible if and only if*

$$\sum_{\tau=1}^T \sum_{j=1}^n d_{j\tau} \leq \sum_{\tau=1}^T \sum_{l=1}^q b_{l\tau}.$$

Proof: The result follows from Lemma 11.2.1 by noticing that $\sum_{i=1}^n x_{ijt} = 1$ for each assignment vector x . \square

Proposition 11.2.4 *If $\mathcal{C} = \emptyset$, the three-level MPSSP is feasible if and only if*

$$\sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} \leq \sum_{\tau=1}^t \sum_{l=1}^q b_{l\tau} \quad t = 1, \dots, T.$$

Proof: The result follows similarly to Proposition 11.2.3. \square

From these feasibility conditions we can deduce that, in contrast to the MPSSP introduced in Chapter 6, the decision problem associated with the feasibility of the three-level MPSSP can be solved in linear time. Moreover, the feasible region of this formulation is of the type of a CCAP. However, the objective function is, in general, not separable in the warehouses.

11.2.3 The LP-relaxation

The LP-relaxation of the three-level MPSSP is obtained by relaxing the constraints $x_{ijt} \in \{0, 1\}$ to nonnegativity constraints. In this section we will show that by fixing the feasible assignments in the optimal solution of the LP-relaxation and arbitrarily assigning the split ones, we obtain a feasible solution for (E). Moreover this solution method is asymptotically optimal. Throughout this section we will assume that all parameters of the three-level MPSSP are bounded. More precisely, there exist nonnegative constants $\underline{A}, \bar{A}, \underline{P}, \bar{P}, \underline{H}, \bar{H}, \underline{D}$, and \bar{D} so that $a_{ijt} \in [\underline{A}, \bar{A}]$, $p_{lit} \in [\underline{P}, \bar{P}]$, $h_{it} \in [\underline{H}, \bar{H}]$, and $d_{jt} \in [\underline{D}, \bar{D}]$ for each $l = 1, \dots, q$, $i = 1, \dots, m$, and $t = 1, \dots, T$.

Under the feasibility conditions given in Propositions 11.2.3 and 11.2.4, the capacity constraints in (E) are not restrictive when assigning customers to warehouses, i.e., any zero-one vector of assignments is a feasible solution for the three-level MPSSP.

Therefore, by fixing the feasible assignments of the LP-relaxation and then arbitrarily assigning the split ones we obtain a feasible solution for (E). Observe that a similar solution method for the MPSSP would fail, in general, to ensure feasibility of the obtained solution.

By construction, the obtained solution vector and the optimal solution vector of the LP-relaxation differ only in the split assignments of the latter one. Similarly as in Lemma 6.4.1 for the MPSSP, the following lemma shows that the number of split assignments in the LP-relaxation of the three-level MPSSP is no more than $mT + qT$. Its proof resembles the proof of Lemma 6.4.1.

Lemma 11.2.5 *For each basic optimal solution for the LP-relaxation of (E), the number of split assignments is at most $mT + qT$.*

Proof: We can rewrite the three-level MPSSP with equality constraints and nonnegativity variables only by introducing slack variables in the production constraints, eliminating the variables x_{ijt} for $i = 1, \dots, m$, $j \in \mathcal{S}$ and $t = 2, \dots, T$, and variables I_{i0} for each $i = 1, \dots, m$. This is a reformulation with, in addition to the assignment constraints, $mT + qT$ equality constraints. In the optimal solution, the number of variables having a nonzero value is no larger than the number of equality constraints. Since there is at least one nonzero assignment variable corresponding to each assignment constraint, and exactly one nonzero assignment variable corresponding to each assignment that is feasible with respect to the integrality constraints of (E), there can be no more than $mT + qT$ assignments that are split. \square

The following result will be needed when showing that the solution method is asymptotically optimal. For notational simplicity, given two feasible solutions for the LP-relaxation of the three-level MPSSP \bar{x} and \tilde{x} , let $\mathcal{A}(\bar{x}, \tilde{x})$ be the set of assignments where they differ, i.e.,

$$\begin{aligned} \mathcal{A}(\bar{x}, \tilde{x}) = & \{j \in \mathcal{S} : \exists i = 1, \dots, m \text{ such that } \bar{x}_{ij1} \neq \tilde{x}_{ij1}\} \\ & \cup \{(j, t) : j \in \mathcal{D}, \exists i = 1, \dots, m \text{ such that } \bar{x}_{ijt} \neq \tilde{x}_{ijt}\}. \end{aligned}$$

Lemma 11.2.6 *Let \bar{x} and \tilde{x} be two feasible solutions for the LP-relaxation of the three-level MPSSP. Then, we have*

$$|H(\bar{x}) - H(\tilde{x})| \leq (\bar{P} - \underline{P} + (T-1)\bar{H}) \cdot T\bar{D} \cdot |\mathcal{A}(\bar{x}, \tilde{x})|.$$

Proof: Without loss of generality we will assume that $H(\bar{x}) \geq H(\tilde{x})$. Moreover, we will also assume that \bar{x} and \tilde{x} just differ in one assignment. In case $|\mathcal{A}(\bar{x}, \tilde{x})| > 1$, we can iteratively repeat the procedure given below. Let $(\tilde{y}, \tilde{I}) \in \mathbb{R}_+^{qmT} \times \mathbb{R}_+^{mT}$ so that

$$H(\tilde{x}) = \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} \tilde{y}_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} \tilde{I}_{it}.$$

Let $(j_1, t_1) \in \mathcal{A}(\bar{x}, \tilde{x})$ be the assignment where those two solutions differ. Similar arguments can be followed if the assignment where they differ corresponds to a static

customer. Suppose that each warehouse supplying (j_1, t_1) in \tilde{x} returns the proportion of the shipment corresponding to (j_1, t_1) to the plants, and the plants increase the corresponding shipments to the warehouses supplying (j_1, t_1) in \tilde{x} so that they are able to satisfy the demand implied by (j_1, t_1) according to \tilde{x} . Let y be the obtained production levels and define $I \in \mathbb{R}_+^{mT}$ so that

$$I_{it} = \sum_{\tau=1}^t \sum_{l=1}^q y_{lit} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} \bar{x}_{ij\tau} - \min_{s=1, \dots, T} \left(\sum_{\tau=1}^s \sum_{l=1}^q y_{lit} - \sum_{\tau=1}^s \sum_{j=1}^n d_{j\tau} \bar{x}_{ij\tau} \right)$$

for each $i = 1, \dots, m$ and $t = 1, \dots, T$ where $I_{i0} = I_{iT}$ for each $i = 1, \dots, m$, if $\mathcal{C} = \{1, \dots, m\}$ and

$$I_{it} = \sum_{\tau=1}^t \sum_{l=1}^q y_{lit} - \sum_{\tau=1}^t \sum_{j=1}^n d_{j\tau} \bar{x}_{ij\tau}$$

for each $i = 1, \dots, m$ and $t = 1, \dots, T$ where $I_{i0} = 0$ for each $i = 1, \dots, m$, if $\mathcal{C} = \emptyset$. By construction, (y, I) belongs to the feasible region of the LP problem defining the value $H(\bar{x})$. Therefore

$$H(\bar{x}) \leq \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} y_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it}.$$

Furthermore, the production levels have been increased by at most \bar{D} and the inventory levels by at most $(T-1)\bar{D}$. If the assignment where they differ corresponds to a static customer, then the production levels are increased by at most $T\bar{D}$ and the inventory levels by at most $T(T-1)\bar{D}$.

Observe that

$$\begin{aligned} |H(\bar{x}) - H(\tilde{x})| &= H(\bar{x}) - H(\tilde{x}) \\ &\leq \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} y_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it} - \\ &\quad - \left(\sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} \tilde{y}_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} \tilde{I}_{it} \right). \end{aligned}$$

Now the result follows easily by using the upper bounds on the production and inventory holding costs. \square

Now we are able to prove asymptotic optimality of this solution method.

Theorem 11.2.7 *If $\mathcal{C} = \{1, \dots, m\}$ or $\mathcal{C} = \emptyset$, fixing the feasible assignments of the optimal solution of the LP-relaxation of (E) and arbitrarily assigning the split ones is a feasible and an asymptotically optimal solution method.*

Proof: The feasibility of this method follows trivially by the feasibility conditions found in Propositions 11.2.3 and 11.2.4. We only need to prove that it is asymptotically optimal. Let x^{LPR} be the optimal solution of the LP-relaxation of (E) and x^{S}

be the vector given by the solution method. It is enough to show that there exists a constant R independent of n so that

$$\left(\sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt}^S + H(x^S) \right) - \left(\sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt}^{\text{LPR}} + H(x^{\text{LPR}}) \right) \leq R.$$

Observe that x^S and x^{LPR} just differ in the split assignments of x^{LPR} . Let B_S be the set of split assignments associated with static customers and B_D be the ones associated with dynamic customers. Then, we have that

$$\begin{aligned} & \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt}^S - \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n a_{ijt} x_{ijt}^{\text{LPR}} = \\ &= \sum_{i=1}^m \sum_{j \in B_S} \left(\sum_{t=1}^T a_{ijt} \right) (x_{ijt}^S - x_{ijt}^{\text{LPR}}) + \sum_{i=1}^m \sum_{(j,t) \in B_D} a_{ijt} (x_{ijt}^S - x_{ijt}^{\text{LPR}}) \\ &\leq \sum_{i=1}^m \sum_{j \in B_S} \left(\sum_{t=1}^T a_{ijt} \right) |x_{ijt}^S - x_{ijt}^{\text{LPR}}| + \sum_{i=1}^m \sum_{(j,t) \in B_D} a_{ijt} |x_{ijt}^S - x_{ijt}^{\text{LPR}}| \\ &\leq \sum_{i=1}^m \sum_{j \in B_S} T \bar{A} |x_{ijt}^S - x_{ijt}^{\text{LPR}}| + \sum_{i=1}^m \sum_{(j,t) \in B_D} \bar{A} |x_{ijt}^S - x_{ijt}^{\text{LPR}}| \\ &\leq \sum_{j \in B_S} 2T \bar{A} + \sum_{(j,t) \in B_D} 2\bar{A} \\ &= 2(|B_S| + |B_D|) T \bar{A}. \end{aligned}$$

Now we would like to bound the difference between $H(x^S)$ and $H(x^{\text{LPR}})$. Since those two vectors just differ in the split assignments of the LP-relaxation, Lemma 11.2.6 ensures that

$$|H(x^S) - H(x^{\text{LPR}})| \leq 2(\bar{P} - \underline{P} + (T-1)\bar{H}) T \bar{D} (|B_S| + |B_D|).$$

Now the result follows by observing that, from Lemma 11.2.5, we have that $|B_S| + |B_D| \leq mT + qT$. \square

11.2.4 Generating experimental data

The stochastic model for the three-level MPSSP is similar to the one given for the MPSSP in Chapter 7. For each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \bar{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed,

i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

Let b_{lt} depend linearly on n , i.e., $b_{lt} = \beta_{lt}n$, for positive constants β_{lt} .

The next results follow easily from Propositions 11.2.3 and 11.2.4.

Corollary 11.2.8 *If $\mathcal{C} = \{1, \dots, m\}$, the three-level MPSSP is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) < \sum_{t=1}^T \sum_{l=1}^q \beta_{lt}$$

and infeasible with probability one if this inequality is reversed.

Corollary 11.2.9 *If $\mathcal{C} = \emptyset$, the three-level MPSSP is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) < \sum_{\tau=1}^t \sum_{l=1}^q \beta_{l\tau} \quad t = 1, \dots, T$$

and infeasible with probability one if at least one of these inequalities is reversed.

11.2.5 A class of greedy heuristics

As for the MPSSP, the class of greedy heuristics proposed for the CCAP applies to the three-level MPSSP. For sake of simplicity, we refer to the static customers and the (customer, period)-pairs for each dynamic customer as tasks, and to the warehouses as agents. Since the capacity constraints do not affect the assignments, each task is assigned to the agent minimizing the pseudo-cost function.

The dual programming problem corresponding to the LP-relaxation of (E) suggests the following family of pseudo-cost functions for the greedy heuristics:

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T (a_{ijt} + \lambda_{it} d_{jt}) & \text{if } \ell = j \in \mathcal{S} \\ a_{ijt} + \lambda_{it} d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$. We will show below that the greedy heuristic is feasible and asymptotically optimal when $\lambda = \lambda^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of (E). Observe that these capacity constraints have been reformulated as \geq -constraints, so that their dual subvector is nonnegative. (Clearly, if the LP-relaxation of (E) is infeasible, so is (E). Therefore, the pseudo-cost function is well-defined.) This family of pseudo-cost functions is similar to the one proposed for the MPSSP.

Based on the feasibility and asymptotic optimality of the solution method given in Section 11.2.3, we will derive similar results for the greedy heuristic with $\lambda =$

λ^* . In those results we will generate the problem instances according to the following stochastic model. Let the random vectors $(\mathbf{D}_j, \mathbf{A}_j, \gamma_j)$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $[\underline{D}, \overline{D}]^T \times [\underline{A}, \overline{A}]^{mT} \times \{0, 1\}$ where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, $\mathbf{A}_j = (\mathbf{A}_{ijt})_{i=1, \dots, m; t=1, \dots, T}$, $(\mathbf{D}_j, \mathbf{A}_j)$ are distributed according to an absolutely continuous probability distribution and \underline{D} , \overline{D} , \underline{A} , and $\overline{A} \in \mathbb{R}_+$. Furthermore, let b_{lt} depend linearly on n , i.e., $b_{lt} = \beta_{lt}n$, for positive constants $\beta_{lt} \in \mathbb{R}_+$. We will assume that $\underline{D} > 0$.

Assumption 11.2.10 *Assume that*

$$\sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) < \sum_{t=1}^T \sum_{l=1}^q \beta_{lt}.$$

Corollary 11.2.11 *If $\mathcal{C} = \{1, \dots, m\}$, then under Assumption 11.2.10 the greedy heuristic is asymptotically feasible and optimal.*

Proof: Observe that the feasibility of the greedy heuristic follows trivially.

Similar as in the proof of Theorem 11.2.7, let x^{LPR} be the optimal solution of the LP-relaxation of (E), B_S be the set of split assignments associated with static customers and B_D be the ones associated with dynamic customers. Following similar steps to Proposition 6.4.3, if the LP-relaxation of (E) is non-degenerate, then the most desirable warehouse for each feasible assignment in the LP-relaxation coincides with the one supplying this assignment in the optimal solution of the LP-relaxation, i.e., for each $j \in \mathcal{S} \setminus B_S$, $x_{ijt}^{\text{LPR}} = 1$ for all $t = 1, \dots, T$ if and only if

$$\sum_{t=1}^T (a_{ijt} + \lambda_{it}^* d_{jt}) = \min_{\ell=1, \dots, m} \sum_{t=1}^T (a_{\ell jt} + \lambda_{\ell t}^* d_{jt})$$

and

$$\sum_{t=1}^T (a_{ijt} + \lambda_{it}^* d_{jt}) < \min_{\ell=1, \dots, m; \ell \neq i} \sum_{t=1}^T (a_{\ell jt} + \lambda_{\ell t}^* d_{jt}),$$

and for each $(j, t) \notin B_D$ so that $j \in \mathcal{D}$, $x_{ijt}^{\text{LPR}} = 1$ if and only if

$$a_{ijt} + \lambda_{it}^* d_{jt} = \min_{\ell=1, \dots, m} (a_{\ell jt} + \lambda_{\ell t}^* d_{jt})$$

and

$$a_{ijt} + \lambda_{it}^* d_{jt} < \min_{\ell=1, \dots, m; \ell \neq i} (a_{\ell jt} + \lambda_{\ell t}^* d_{jt}).$$

As Proposition 8.3.2, we have that the LP-relaxation of (E) is non-degenerate with probability one, under the proposed stochastic model since, for each customer, the vector of cost and requirement parameters is distributed according to an absolutely continuous probability distribution. Therefore, the solution given by the greedy heuristic and the optimal solution of the LP-relaxation coincides for all the feasible

assignments of the later one with probability one. Now the result follows similarly to Theorem 11.2.7. \square

A similar result is given for the acyclic case.

Assumption 11.2.12 *Assume that*

$$\sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) < \sum_{\tau=1}^t \sum_{l=1}^q \beta_{l\tau} \quad t = 1, \dots, T.$$

Corollary 11.2.13 *If $\mathcal{C} = \emptyset$, then under Assumption 11.2.12 the greedy heuristic is asymptotically feasible and optimal.*

Proof: It follows using the same arguments as in Corollary 11.2.11. \square

11.3 Throughput capacity constraints

11.3.1 A reformulation

In this section we will investigate the addition of throughput capacity constraints at the warehouses to the three-level MPSSP. Recall that those constraints are of the form

$$\sum_{j=1}^n d_{jt} x_{ijt} \leq r_{it} \quad i = 1, \dots, m; t = 1, \dots, T$$

where r_{it} is the throughput capacity at warehouse i in period t . A similar reformulation to the one given in Section 11.2.2 for the three-level MPSSP holds where we have to substitute the function H by \tilde{H} defined as

$$\tilde{H}(x) = \begin{cases} H(x) & \text{if } \sum_{j=1}^n d_{jt} x_{ijt} \leq r_{it} \text{ for } i = 1, \dots, m; t = 1, \dots, T \\ \infty & \text{otherwise} \end{cases}$$

for each $x \in \mathbb{R}^{mnT}$.

The domain of the function \tilde{H} can be easily derived from the domain of the function H . We will present the two extreme cases of the set \mathcal{C} separately.

Corollary 11.3.1 *If $\mathcal{C} = \{1, \dots, m\}$, the domain of the function \tilde{H} is equal to*

$$\left\{ x \in \mathbb{R}_+^{mnT} : \sum_{t=1}^T \sum_{i=1}^m \sum_{j=1}^n d_{jt} x_{ijt} \leq \sum_{t=1}^T \sum_{l=1}^q b_{lt}, \right. \\ \left. \sum_{j=1}^n d_{jt} x_{ijt} \leq r_{it}, i = 1, \dots, m; t = 1, \dots, T \right\}.$$

Corollary 11.3.2 *If $\mathcal{C} = \emptyset$, the domain of the function \tilde{H} is equal to*

$$\left\{ z \in \mathbb{R}_+^{nT} : \sum_{\tau=1}^t \sum_{i=1}^m \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{\tau=1}^t \sum_{l=1}^q b_{l\tau}, t = 1, \dots, T, \right. \\ \left. \sum_{j=1}^n d_{jt} x_{ijt} \leq r_{it}, i = 1, \dots, m; t = 1, \dots, T \right\}.$$

Note that the throughput capacity constraints are also present in the feasible region of the reformulation given above. Several conclusions can be drawn from Corollaries 11.3.1 and 11.3.2. First, as for the MPSSP introduced in Chapter 6, the decision problem associated with the feasibility of this model is an \mathcal{NP} -Complete problem, in contrast to the three-level MPSSP. Second, the feasible region of the reformulation of the three-level MPSSP is of the type of a CCAP. Therefore, we can make use of the stochastic model for the CCAP and the implicit feasibility condition to ensure asymptotic feasibility of the problem instances generated by the stochastic model with probability one (as n goes to infinity), as well of the class of greedy heuristics.

11.3.2 Generating experimental data

The stochastic model is similar to the one given for the corresponding two-level model in Section 10.2.3. For each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \overline{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

Let b_{lt} depend linearly on n , i.e., $b_{lt} = \beta_{lt}n$, for positive constants β_{lt} . Let r_{it} also depend linearly on n , i.e., $r_{it} = \rho_{it}n$ for positive constants ρ_{it} .

The following corollaries show explicit feasibility conditions for several variants of the three-level MPSSP with throughput capacity constraints.

Corollary 11.3.3 *If $\mathcal{C} = \{1, \dots, m\}$ and $\mathcal{S} = \emptyset$, the three-level MPSSP with throughput capacity constraints is feasible with probability one, as $n \rightarrow \infty$, if*

$$\sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) < \sum_{t=1}^T \sum_{l=1}^q \beta_{lt} \\ \mathcal{E}(\mathbf{D}_{1t}) < \sum_{i=1}^m \rho_{it} \quad t = 1, \dots, T$$

and infeasible with probability one if at least one of these inequalities is reversed.

Proof: When all warehouses have a cyclic demand pattern and all customers are dynamic, the function \tilde{H} at vectors corresponding to assignments is finite if and only if the throughput capacity constraints hold, see Lemma 11.2.1. (Observe that $\tilde{H}(x) = H(x)$ for each vector x satisfying the throughput capacity constraints.) Now the result follows similarly to Theorem 7.4.2. \square

A similar result can be found for the static variant of this model.

Corollary 11.3.4 *If $\mathcal{C} = \emptyset$ and $\mathcal{S} = \emptyset$, the three-level MPSSP with throughput capacity constraints is feasible with probability one, as $n \rightarrow \infty$, if*

$$\begin{aligned} \sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) &< \sum_{\tau=1}^t \sum_{l=1}^q \beta_{l\tau} & t = 1, \dots, T \\ \mathcal{E}(\mathbf{D}_{1t}) &< \sum_{i=1}^m \rho_{it} & t = 1, \dots, T \end{aligned}$$

and infeasible with probability one if at least one of these inequalities is reversed.

Proof: The result follows similarly to Corollary 11.3.3 by observing that the feasible region is formed by the Cartesian product of the feasible regions of T GAP's with agent-independent requirements. \square

As for the original MPSSP, the reformulation of the static and seasonal MPSSP with throughput capacity constraints is a nonlinear GAP with agent-independent requirements. This allows us to derive explicit feasibility conditions for it.

Corollary 11.3.5 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all the customers have the same demand pattern, the three-level MPSSP with throughput capacity constraints is feasible with probability one, as $n \rightarrow \infty$, if*

$$\begin{aligned} \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) &< \sum_{t=1}^T \sum_{l=1}^q \beta_{lt} \\ \mathcal{E}(\mathbf{D}_{11}) &< \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right) \end{aligned}$$

and infeasible with probability one if at least one of these inequalities is reversed.

Corollary 11.3.6 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all the customers have the same demand pattern, the three-level MPSSP with throughput capacity constraints is feasible with probability one, as $n \rightarrow \infty$, if*

$$\begin{aligned} \sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) &< \sum_{\tau=1}^t \sum_{l=1}^q \beta_{l\tau} & t = 1, \dots, T \\ \mathcal{E}(\mathbf{D}_{11}) &< \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right) \end{aligned}$$

and infeasible with probability one if at least one of these inequalities is reversed.

11.3.3 A class of greedy heuristics

We can solve the problem with the class of greedy heuristics for the CCAP. For this model, checking feasibility of an assignment implies showing that, for a given vector x , the feasible region of the LP problem defining the value $\tilde{H}(x)$ is nonempty.

The dual programming problem corresponding to the LP-relaxation of the three-level MPSSP with throughput capacity constraints suggests the following family of pseudo-cost functions for the greedy heuristics

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T (a_{ijt} + (\lambda_{it} + \xi_{it})d_{jt}) & \text{if } \ell = j \in \mathcal{S} \\ a_{ijt} + (\lambda_{it} + \xi_{it})d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$ and $\xi = (\xi_{it}) \in \mathbb{R}_+^{mT}$. As for the corresponding two-level model, we may expect good results when $\lambda = \lambda^*$ and $\xi = \xi^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of the three-level MPSSP with throughput capacity constraints, and similarly, ξ^* represents the optimal dual subvector corresponding to the throughput capacity constraints. Observe that both types of capacity constraints have been reformulated as \geq -constraints, so that their dual subvectors are nonnegative. (Clearly, if the LP-relaxation is infeasible, so is the three-level MPSSP with throughput capacity constraints.) Again, the family of pseudo-cost functions is similar to the one proposed for the MPSSP with throughput capacity constraints.

The following asymptotic results follow from the ones found for the GAP with agent-independent requirements and Lemma 11.2.6. We assume that the problem instances are generated according to the following stochastic model. Let the random vectors $(\mathbf{D}_j, \mathbf{A}_j, \gamma_j)$ ($j = 1, \dots, n$) be i.i.d. in the bounded set $[\underline{D}, \overline{D}]^T \times [\underline{A}, \overline{A}]^{mT} \times \{0, 1\}$, $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, $\mathbf{A}_j = (\mathbf{A}_{ijt})_{i=1, \dots, m; t=1, \dots, T}$, $(\mathbf{D}_j, \mathbf{A}_j)$ are distributed according to an absolutely continuous probability distribution and \underline{D} , \overline{D} , \underline{A} , and $\overline{A} \in \mathbb{R}_+$. Furthermore, let b_{lt} depend linearly on n , i.e., $b_{lt} = \beta_{lt}n$, for positive constants $\beta_{lt} \in \mathbb{R}_+$. Let r_{it} also depend linearly on n , i.e., $r_{it} = \rho_{it}n$ for positive constants ρ_{it} . We will assume that $\underline{D} > 0$.

In a similar way as for the MPSSP with throughput capacity constraints, the greedy heuristic with $\lambda = \lambda^*$ and $\xi = \xi^*$ is asymptotically feasible and optimal when all customers are static and have the same demand pattern. The proof of this result is based on showing that the reformulation is a GAP with a nonlinear function (see Corollary 11.3.3), and it also uses Lemma 11.2.6 where we show that a solution vector close to the optimal solution of the LP-relaxation is also close in objective values. We will present the two extreme cases of the set \mathcal{C} separately.

Assumption 11.3.7 *Assume that*

$$\sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) < \sum_{t=1}^T \sum_{l=1}^q \beta_{lt}$$

$$\mathcal{E}(\mathbf{D}_{1t}) < \sum_{i=1}^m \rho_{it} \quad t = 1, \dots, T.$$

Corollary 11.3.8 *If $\mathcal{C} = \{1, \dots, m\}$ and $\mathcal{S} = \emptyset$, then under Assumption 11.3.7 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: The feasible region of the CCAP formulation of the three-level MPSSP with throughput capacity constraints is defined by the assignment and the throughput constraints. Now the result follows in a similar way as Theorem 8.5.7 by observing that a solution vector close to the optimal solution of the LP-relaxation is also close in objective values. \square

Assumption 11.3.9 *Assume that*

$$\begin{aligned} \sum_{t=1}^T \mathcal{E}(\mathbf{D}_{1t}) &< \sum_{t=1}^T \sum_{l=1}^q \beta_{lt} \\ \mathcal{E}(\mathbf{D}_{11}) &< \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right). \end{aligned}$$

Corollary 11.3.10 *If $\mathcal{C} = \{1, \dots, m\}$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 11.3.9 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: The result follows similarly to Corollary 11.3.8. \square

We will make a similar assumption for the acyclic case.

Assumption 11.3.11 *Assume that*

$$\begin{aligned} \sum_{\tau=1}^t \mathcal{E}(\mathbf{D}_{1\tau}) &< \sum_{\tau=1}^t \sum_{l=1}^q \beta_{l\tau} \quad t = 1, \dots, T \\ \mathcal{E}(\mathbf{D}_{11}) &< \sum_{i=1}^m \min_{t=1, \dots, T} \left(\frac{\rho_{it}}{\sigma_t} \right). \end{aligned}$$

Corollary 11.3.12 *If $\mathcal{C} = \emptyset$, $\mathcal{S} = \{1, \dots, n\}$, and all customers have the same demand pattern, then under Assumption 11.3.11 the greedy heuristic is asymptotically feasible and optimal with probability one.*

Proof: The result follows similarly to Corollary 11.3.8. \square

We conjecture that this greedy heuristic is asymptotically feasible and optimal for other variants of the three-level MPSSP with throughput capacity constraints.

11.4 Physical capacity constraints

11.4.1 A reformulation

In this section we will analyze the addition of physical capacity constraints at each warehouse to the three-level MPSSP. Recall that those constraints are of the form

$$I_{it} \leq \bar{I}_{it} \quad i = 1, \dots, m; t = 1, \dots, T$$

where \bar{I}_{it} is the physical capacity at warehouse i in period t . A similar reformulation to the one given in Section 11.2.2 for the three-level MPSSP holds where the function H must be substituted by \bar{H} , where $\bar{H}(x)$, $x \in \mathbb{R}_+^{mnT}$, is defined as the optimal solution value of the following linear programming problem

$$\text{minimize } \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} y_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it}$$

subject to

$$\begin{aligned} I_{i,t-1} - I_{it} + \sum_{l=1}^q y_{lit} &= \sum_{j=1}^n d_{jt} x_{ijt} & i = 1, \dots, m; t = 1, \dots, T \\ \sum_{i=1}^m y_{lit} &\leq b_{lt} & l = 1, \dots, q; t = 1, \dots, T \\ I_{i0} &= I_{iT} 1_{\{i \in C\}} & i = 1, \dots, m \\ I_{it} &\leq \bar{I}_{it} & i = 1, \dots, m; t = 1, \dots, T \\ y_{lit} &\geq 0 & l = 1, \dots, q; i = 1, \dots, m; t = 1, \dots, T \\ I_{it} &\geq 0 & i = 1, \dots, m; t = 1, \dots, T. \end{aligned}$$

As a difference with the three-level MPSSP analyzed until now, in general, we have not yet been able to characterize the domain of the function \bar{H} , i.e., the set of vectors $x \in \mathbb{R}^{mnT}$ where the function \bar{H} is finite. Observe that warehouses can receive shipments from any plant. So, we might expect that the feasibility conditions are just the aggregation over the set of all warehouses of the ones found for \bar{H}_i in Section 10.3.2. They would read for the cyclic case as follows:

$$\begin{aligned} \sum_{i=1}^m \sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} x_{ij[t]} &\leq \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{l[t]} + \sum_{i=1}^m \bar{I}_{i[\underline{t}-1]} \\ &\underline{t} = 1, \dots, T; r = 1, \dots, T-1 \\ \sum_{t=1}^T \sum_{j=1}^n d_{jt} &\leq \sum_{l=1}^q \sum_{t=1}^T b_{lt}, \end{aligned}$$

and in the acyclic case,

$$\sum_{i=1}^m \sum_{\tau=\underline{t}}^{\bar{t}} \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{l=1}^q \sum_{\tau=\underline{t}}^{\bar{t}} b_{l\tau} + \sum_{i=1}^m \bar{I}_{i,\underline{t}-1} \quad (11.15)$$

$$\underline{t} = 1, \dots, T; \bar{t} = \underline{t}, \dots, T.$$

We will see below that those conditions are necessary. However, the following counterexample shows that they are, in general, not sufficient. Consider just one plant, two warehouses with acyclic inventory pattern and a planning horizon of length two. Let the capacities of the plant be equal to 200 units in the first period and 30 units in the second one. The first warehouse requires 50 units in the first period and 60 units in the second period, moreover, its maximal inventory level in the first period is equal to 10 units. Similarly for the second warehouse, the demand in the first period is equal to 60 units and in the second period 40 units, and its maximal inventory level in the first period is equal to 60 units. Observe that the conditions given in (11.15) are satisfied however the problem is infeasible. To supply the first warehouse in the second period we need at least 30 units of inventory but on the other hand the maximal inventory level is 10 units. This suggests that this type of constraints must be also satisfied for each subset of warehouses. It is easy to prove that those conditions are necessary. We will study the cyclic and the acyclic case separately.

Lemma 11.4.1 *If $\mathcal{C} = \{1, \dots, m\}$, the following conditions are necessary to ensure that $\bar{H}(x)$ is finite:*

$$\begin{aligned} \sum_{i \in \mathcal{M}} \sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} x_{ij[t]} &\leq \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{l[t]} + \sum_{i \in \mathcal{M}} \bar{I}_{i[\underline{t}-1]} \\ \mathcal{M} &\subseteq \{1, \dots, m\}; \underline{t} = 1, \dots, T; r = 1, \dots, T-1 \\ \sum_{t=1}^T \sum_{j=1}^n d_{jt} &\leq \sum_{l=1}^q \sum_{t=1}^T b_{lt}. \end{aligned}$$

Proof: By aggregating the r consecutive balance constraints following the \underline{t} -th one, $\underline{t} = 1, \dots, T$, and $r = 1, \dots, T-1$ over the warehouses in $\mathcal{M} \subseteq \{1, \dots, m\}$, we obtain

$$\sum_{i \in \mathcal{M}} \sum_{t=\underline{t}+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} x_{ij[t]} + \sum_{i \in \mathcal{M}} I_{i[\bar{t}]} = \sum_{i \in \mathcal{M}} \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} y_{li[t]} + \sum_{i \in \mathcal{M}} I_{i[\underline{t}-1]}.$$

Now the desired inequality follows by observing that $\sum_{i \in \mathcal{M}} I_{i[\bar{t}]} \geq 0$,

$$\sum_{i \in \mathcal{M}} \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} y_{li[t]} \leq \sum_{i=1}^m \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} y_{li[t]} \leq \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{l[t]},$$

and finally using the upper bound on $I_{i[\underline{t}-1]}$. The second type of inequalities follows by simply aggregating the T balance constraints over the warehouses in \mathcal{M} , and then again using the bound on the production quantities. \square

Lemma 11.4.2 *If $\mathcal{C} = \emptyset$, the following conditions are sufficient to ensure that $\overline{H}(x)$ is finite:*

$$\sum_{i \in \mathcal{M}} \sum_{\tau=\underline{t}}^{\bar{t}} \sum_{j=1}^n d_{j\tau} x_{ij\tau} \leq \sum_{l=1}^q \sum_{\tau=\underline{t}}^{\bar{t}} b_{l\tau} + \sum_{i \in \mathcal{M}} \bar{I}_{i,\underline{t}-1}$$

$$\mathcal{M} \subseteq \{1, \dots, m\}; \underline{t} = 1, \dots, T; \bar{t} = \underline{t}, \dots, T.$$

Proof: Similar to the proof of Lemma 11.4.1. □

We conjecture that these constraints are also sufficient.

Conjecture 11.4.3 *If $\mathcal{C} = \{1, \dots, m\}$, the conditions given in Lemma 11.4.1 are sufficient to ensure that $\overline{H}(x)$ is finite.*

We have a similar conjecture for the acyclic case.

Conjecture 11.4.4 *If $\mathcal{C} = \emptyset$, the conditions given in Lemma 11.4.2 are sufficient to ensure that $\overline{H}(x)$ is finite.*

We will verify Conjecture 11.4.4 for a variant of the three-level MPSSP with physical capacity constraints with two warehouses and two periods.

Theorem 11.4.5 *If $\mathcal{C} = \emptyset$, $m = 2$, and $T = 2$, then the conditions given in Lemma 11.4.2 are necessary and sufficient to ensure that $\overline{H}(x)$ is finite.*

Proof: The necessity of these conditions was proved in Lemma 11.4.2. Thus, it remains to prove the sufficiency. Without loss of generality, we can assume that there is just one plant supplying to all warehouses. For sake of simplicity, let $u_{it} = \sum_{j=1}^n d_{jt} x_{ijt}$.

If $u_{12} + u_{22} \leq b_{12}$, the vector (y, I) defined as $y_{it} = u_{it}$ and $I_{it} = 0$ for each $i = 1, 2$ and $t = 1, 2$ belongs to the feasible region of the LP problem associated with $\overline{H}(x)$ and then $\overline{H}(x)$ is finite.

Now assume that $u_{12} + u_{22} > b_{12}$. To prove that $\overline{H}(x)$ is finite, it is enough to find $p_i \in [0, u_{i2}]$, $i = 1, 2$, so that

$$p_1 + p_2 = b_{12} \tag{11.16}$$

$$u_{12} - p_1 \leq \bar{I}_{11} \tag{11.17}$$

$$u_{22} - p_2 \leq \bar{I}_{21} \tag{11.18}$$

$$u_{11} + u_{21} + u_{12} - p_1 + u_{22} - p_2 \leq b_{11}. \tag{11.19}$$

Then, the vector (y, I) , where $y_{i1} = u_{i1} + u_{i2} - p_i$, $y_{i2} = p_i$, $I_{i1} = u_{i2} - p_i$ and $I_{i2} = 0$ for $i = 1, 2$, belongs to the feasible region of the LP problem defining the value $\overline{H}(x)$, and the result follows. Substituting the equation (11.16) in the inequality (11.19), we obtain one of the conditions in Conjecture 11.4.4. Therefore, condition (11.19) is always satisfied if condition (11.16) holds.

Observe that the interval $[0, u_{22}]$ has nonempty intersection with the interval $[u_{22} - \bar{I}_{21}, b_{12} + \bar{I}_{11} - u_{12}]$. This follows easily since $u_{22} \geq u_{22} - \bar{I}_{21}$ by the nonnegativity of \bar{I}_{21} and $b_{12} + \bar{I}_{11} - u_{12} \geq 0$ by one of the conditions in the conjecture. Then,

$$[\max\{0, u_{22} - \bar{I}_{21}\}, \min\{u_{22}, b_{12} + \bar{I}_{11} - u_{12}\}] \neq \emptyset.$$

Now we will prove that this interval has nonempty intersection with the interval $[b_{12} - u_{12}, b_{12}]$. This is again easy to prove. By the nonnegativity of b_{12} and one of the conditions in the conjecture we have that $\max\{0, u_{22} - \bar{I}_{21}\} \leq b_{12}$. Moreover, using that $u_{12} + u_{22} > b_{12}$ and the nonnegativity of \bar{I}_{11} , it follows that $b_{12} - u_{12} \leq \min\{u_{22}, b_{12} + \bar{I}_{11} - u_{12}\}$. Therefore, under the conditions of Conjecture 11.4.4, it holds that

$$[\max\{0, u_{22} - \bar{I}_{21}\}, \min\{u_{22}, b_{12} + \bar{I}_{11} - u_{12}\}] \cap [b_{12} - u_{12}, b_{12}] \neq \emptyset.$$

Moreover, for each p_2 in that interval, we can easily prove that $p_1 = b_{12} - p_2$ and p_2 satisfy conditions (11.16)-(11.19). Then, the result follows. \square

11.4.2 Generating experimental data

We can still define a stochastic model for the three-level MPSSP with physical capacity constraints based on the one introduced for the CCAP. For each customer $j = 1, \dots, n$, let (\mathbf{D}_j, γ_j) be i.i.d. random vectors in $[\underline{D}, \bar{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

Let b_{lt} depend linearly on n , i.e., $b_{lt} = \beta_{lt}n$, for positive constants β_{lt} . Furthermore, we will assume that \bar{I}_{it} depends linearly on n , i.e., $\bar{I}_{it} = \eta_{it}n$ for positive constants η_{it} . However, in general, we cannot make use of the implicit feasibility condition found for the CCAP since we could not characterize the domain of the function \bar{H} as a set of linear constraints.

11.4.3 A class of greedy heuristics

The class of greedy heuristics for the CCAP applies to the three-level MPSSP with physical capacity constraints. As for the corresponding two-level model, checking feasibility of an assignment implies showing that, for a given vector x , the feasible region of the LP problem defining the value $\bar{H}(x)$ is nonempty. Of course, this is computational harder than just checking a couple of inequalities as in the MPSSP's discussed until now.

The dual programming problem corresponding to the LP-relaxation of the three-level MPSSP with physical capacity constraints suggests the following family of pseudo-cost functions for the greedy heuristics:

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T (a_{ijt} + \lambda_{it}d_{jt}) & \text{if } \ell = j \in \mathcal{S} \\ a_{ijt} + \lambda_{it}d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$. As for the corresponding two-level model, we may expect good results when $\lambda = \lambda^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of the three-level MPSSP with physical capacity constraints. Again the production capacity constraints have been reformulated as \geq -constraints, so that its dual subvector is non-negative. (Clearly, if the LP-relaxation is infeasible, so is the three-level MPSSP with physical capacity constraints. Therefore, the pseudo-cost function is well-defined.) Again, this family of pseudo-cost functions is similar to the one proposed for the MPSSP with physical capacity constraints.

We conjecture that this greedy heuristic is asymptotically feasible and optimal for some variants of the three-level MPSSP with physical capacity constraints.

11.5 Perishability constraints

11.5.1 A reformulation

In this section we will analyze the addition of perishability constraints at each warehouse to the three-level MPSSP. We will assume that the shelf-life of the product is equal to k periods, $k = 1, \dots, T - 2$. Recall that those constraints are of the form

$$\sum_{\tau=t+1}^{t+k} \sum_{j=1}^n 1_{\{i \in \mathcal{C} \vee \tau \leq T\}} d_{j[\tau]} x_{ij[\tau]} \geq I_{it} \quad t = 1, \dots, T.$$

A similar reformulation to the one given in Section 11.2.2 for the three-level MPSSP holds where we have to substitute the function H by H^k , where $H^k(x)$, $x \in \mathbb{R}_+^{mnT}$, is defined as the optimal solution value of the following linear programming problem

$$\text{minimize } \sum_{t=1}^T \sum_{l=1}^q \sum_{i=1}^m p_{lit} y_{lit} + \sum_{t=1}^T \sum_{i=1}^m h_{it} I_{it}$$

subject to

$$\begin{aligned} I_{i,t-1} - I_{it} + \sum_{l=1}^q y_{lit} &= \sum_{j=1}^n d_{jt} x_{ijt} & i = 1, \dots, m; t = 1, \dots, T \\ \sum_{i=1}^m y_{lit} &\leq b_{lt} & l = 1, \dots, q; t = 1, \dots, T \\ I_{it} &\leq \sum_{\tau=t+1}^{t+k} \sum_{j=1}^n 1_{\{i \in \mathcal{C} \vee \tau \leq T\}} d_{j[\tau]} x_{ij[\tau]} & t = 1, \dots, T \\ I_{i0} &= I_{iT} 1_{\{i \in \mathcal{C}\}} & i = 1, \dots, m \\ y_{lit} &\geq 0 & l = 1, \dots, q; i = 1, \dots, m; t = 1, \dots, T \\ I_{it} &\geq 0 & i = 1, \dots, m; t = 1, \dots, T. \end{aligned}$$

As for the three-level MPSSP with physical capacity constraints, we have not yet been able to characterize the domain of the function H^k . We have again some necessary conditions to ensure that H^k is finite and we conjecture that these conditions are also sufficient.

Lemma 11.5.1 *If $\mathcal{C} = \{1, \dots, m\}$, the following conditions are necessary to ensure that $H^k(x)$ is finite:*

$$\begin{aligned} \sum_{i \in \mathcal{M}} \sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{j[t]} x_{ij[t]} &\leq \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{l[t]} \\ \mathcal{M} &\subseteq \{1, \dots, m\}; \underline{t} = 1, \dots, T; r = k+1, \dots, T-1 \\ \sum_{t=1}^T \sum_{j=1}^n d_{jt} &\leq \sum_{l=1}^q \sum_{t=1}^T b_{lt}. \end{aligned}$$

Proof: Similar to the proof of Lemma 11.4.1. □

We have a similar lemma for the acyclic case.

Lemma 11.5.2 *If $\mathcal{C} = \emptyset$, the following conditions are necessary to ensure that $H^k(x)$ is finite:*

$$\begin{aligned} \sum_{i \in \mathcal{M}} \sum_{t=\underline{t}+k+1}^{\underline{t}+r} \sum_{j=1}^n d_{j\tau} x_{ij\tau} &\leq \sum_{l=1}^q \sum_{t=\underline{t}+1}^{\underline{t}+r} b_{l\tau} \\ \mathcal{M} &\subseteq \{1, \dots, m\}; \underline{t} = 1, \dots, T; r = k+1, \dots, T-\underline{t} \\ \sum_{t=1}^{\underline{t}} \sum_{j=1}^n d_{jt} z_{jt} &\leq \sum_{l=1}^q \sum_{t=1}^{\underline{t}} b_{lt} \quad \underline{t} = 1, \dots, T. \end{aligned}$$

Proof: Similar to the proof of Lemma 11.4.1. □

In the following we conjecture that those conditions are also sufficient.

Conjecture 11.5.3 *If $\mathcal{C} = \{1, \dots, m\}$, the conditions given in Lemma 11.5.1 are sufficient to ensure that $H^k(x)$ is finite.*

Conjecture 11.5.4 *If $\mathcal{C} = \emptyset$, the conditions given in Lemma 11.5.2 are sufficient to ensure that $H^k(x)$ is finite.*

11.5.2 Generating experimental data

We can define a stochastic model for this problem based on the one proposed for the CCAP. For each customer $j = 1, \dots, n$, let $(\mathbf{D}_j, \boldsymbol{\gamma}_j)$ be i.i.d. random vectors in

$[\underline{D}, \overline{D}]^T \times \{0, 1\}$, where $\mathbf{D}_j = (\mathbf{D}_{jt})_{t=1, \dots, T}$, γ_j is Bernoulli-distributed, i.e., $\gamma_j \sim \text{Be}(\pi)$, with $\pi \in [0, 1]$, and

$$\gamma_j = \begin{cases} 0 & \text{if } j \in \mathcal{S} \\ 1 & \text{if } j \in \mathcal{D}. \end{cases}$$

Let b_{lt} depend linearly on n , i.e., $b_{lt} = \beta_{lt}n$, for positive constants β_{lt} . Since we were not able to characterize the domain of the function H^k by linear constraints, we cannot make use of the implicit feasibility condition found for the CCAP.

11.5.3 A class of greedy heuristics

As in previous models, we can solve the problem with the class of greedy heuristics for the CCAP. Checking feasibility of an assignment is equivalent to proving that, for a given vector x , the feasible region of the LP problem defining the value $H^k(x)$ is nonempty.

The dual programming problem corresponding to the LP-relaxation of the three-level MPSSP with perishability constraints suggests the following family of pseudo-cost functions for the greedy heuristics:

$$f(i, \ell) = \begin{cases} \sum_{t=1}^T \left(a_{ijt} + \left(\lambda_{it} - \sum_{\ell=1}^k 1_{\{i \in C \vee \ell < t\}} \alpha_{i[t-\ell]} \right) d_{jt} \right) & \text{if } \ell = j \in \mathcal{S} \\ a_{ijt} + \left(\lambda_{it} - \sum_{\ell=1}^k 1_{\{i \in C \vee \ell < t\}} \alpha_{i[t-\ell]} \right) d_{jt} & \text{if } \ell = (j, t); j \in \mathcal{D} \text{ and } t = 1, \dots, T \end{cases}$$

where $\lambda = (\lambda_{it}) \in \mathbb{R}_+^{mT}$ and $\alpha = (\alpha_{it}) \in \mathbb{R}_+^{mT}$. We may expect good results when $\lambda = \lambda^*$ and $\alpha = \alpha^*$, where λ^* represents the optimal dual subvector corresponding to the production capacity constraints in the LP-relaxation of the three-level MPSSP with perishability constraints, and similarly, α^* represents the optimal dual subvector corresponding to perishability constraints. Again, both types of capacity constraints have been reformulated as \geq -constraints, so that their dual subvectors are nonnegative. (Clearly, if the LP-relaxation is infeasible, so is the three-level MPSSP with perishability constraints.) This family of pseudo-cost functions is similar to the one proposed for the MPSSP with perishability constraints.

We conjecture that this greedy heuristic is asymptotically feasible and optimal for some variants of the three-level MPSSP with perishability constraints.

11.6 Numerical illustrations

In this section we will illustrate the behaviour of the greedy heuristic given in Section 11.3.3 for a variant of the three-level MPSSP with throughput capacity constraints on a set of randomly generated test problems. We have considered all the warehouses with a cyclic inventory pattern and all the customers are dynamic. For each problem instance, we generate a set of customers, a set of warehouses, and a set of plants uniformly in the square $[0, 10]^2$. For customer j ($j = 1, \dots, n$), we generate a random demand \mathbf{D}_{jt} in period t ($t = 1, \dots, T$) from the uniform distribution

on $[5\sigma_t, 25\sigma_t]$, where the vector σ contains seasonal factors, which we have chosen to be $\sigma = (\frac{1}{2}, \frac{3}{4}, 1, 1, \frac{3}{4}, \frac{1}{2})^\top$. The production costs are assumed to be equal to the distance, i.e., $p_{lit} = \text{dist}_{li}$, where dist_{li} denotes the Euclidean distance between plant l and warehouse i . The assignment costs are assumed to be proportional to demand and distance, i.e., $a_{ijt} = d_{jt} \cdot \text{dist}_{ij}$, where dist_{ij} denotes the Euclidean distance between warehouse i and customer j . Finally, we generate inventory holding costs \mathbf{H}_{it} uniformly from $[10, 30]$.

We have chosen the capacities equal to $b_{lt} = \frac{1}{q} \cdot \beta \cdot n$ and $r_{it} = \frac{1}{m} \cdot \rho \cdot n$, where

$$\begin{aligned}\beta &= \delta \cdot 15 \cdot \max_{t=1, \dots, T} \left(\frac{1}{t} \sum_{\tau=1}^t \sigma_\tau \right) \\ \rho &= \delta \cdot 15 \cdot \max_{t=1, \dots, T} \sigma_t.\end{aligned}$$

Using Corollary 11.3.4 we can easily prove that the problem instances generated by this probabilistic model are asymptotically feasible with probability one (as n goes to infinity) if $\delta > 1$, and infeasible with probability one (again as n goes to infinity) if $\delta < 1$. To account for the asymptotic nature of this feasibility guarantee, we have set $\delta = 1.1$ to obtain feasible problem instances for finite n .

We have fixed the number of plants at $q = 3$, the number of warehouses at $m = 5$, and the number of periods at $T = 6$. We let the number of customers vary from $n = 50$ until $n = 500$ in increments of 50 customers. For each size of the problem we have generated 50 problem instances. All the runs were performed on a PC with a 350 MHz Pentium II processor and 128 MB RAM. All LP-relaxations were solved using CPLEX 6.5 [33].

Table 11.1 illustrates the behaviour of this greedy heuristic. Clearly, n denotes the number of customers. The table shows the number of problem instances for which the LP-relaxation was feasible, as well as the number of problem instances for which the greedy heuristic found a feasible solution. In addition, the time needed to solve the LP-relaxation, as well as the total time needed to find the greedy heuristic solution (i.e., including the time needed to solve the LP-relaxation) is shown. Finally, an upper bound on the average error of the greedy heuristic solution is shown, as measured by the relative deviation of the greedy heuristic solution value from the optimal value of the LP-relaxation. This average was calculated only using the problem instances where the greedy heuristic found a feasible solution.

Although we cannot guarantee that the greedy heuristic will always find a feasible solution (recall that even to determine whether a particular problem instance of the three-level MPSSP with throughput capacity constraints is feasible is an \mathcal{NP} -Complete problem), a feasible solution was always found for problem instances with at least 150 customers. Note that feasibility of the LP-relaxation does not imply feasibility of the three-level MPSSP with throughput capacity constraints, so that the inability of the greedy heuristic to find a feasible solution could be caused by infeasibility of the problem instance, even when the LP-relaxation is feasible.

Except for the smallest class of problem instances, the average error was always well below 1%. Moreover, the fact that the average error decreases as the number of

<i>n</i>	LP		heuristic		
	# feasible	time (sec.)	# feasible	time (sec.)	error (%)
50	42	0.17	35	0.21	1.26
100	48	0.42	47	0.46	0.59
150	50	0.68	50	0.74	0.42
200	50	1.20	50	1.26	0.28
250	50	1.80	50	1.87	0.24
300	50	2.34	50	2.42	0.28
350	50	2.90	50	2.98	0.16
400	50	3.37	50	3.46	0.17
450	50	3.91	50	4.00	0.12
500	50	4.83	50	4.93	0.12

Table 11.1: Greedy heuristic for the three-level MPSSP with throughput capacity constraints

customers increases supports our conjecture that the greedy heuristic is asymptotically optimal.

In addition to using the greedy heuristic, we have also used the MIP solver of CPLEX to try to solve the problems to optimality for the two smallest problem sizes. The procedure was cut after 30 minutes, which happened for 12 problem instances with 50 customers, and for 29 problem instances with 100 customers. For the problem instances with 50 customers, the average time spent by CPLEX was 525 seconds (and 122 seconds when we disregard the most difficult problem instances where the optimal solution was not found within 30 minutes). For the problem instances with 100 customers, these numbers were 1200 and 362, respectively. Comparing these to the times spent by the greedy heuristic, we conclude that the greedy heuristic is a very effective way of finding a high quality solution with little effort.

Chapter 12

Summary and concluding remarks

In this thesis we have studied a class of optimization models that gives an insight to companies when looking for opportunities for improving the efficiency of their logistics distribution networks in a dynamic environment. Several arguments have motivated this research. Firstly, the supply chain is, nowadays, subject to more dynamics than in the past. This forces companies to regularly reconsider their logistics distribution networks. Secondly, due to the complexity of the evaluation of the design of logistics distribution networks, most of the optimization models proposed in the literature are static in nature, therefore they prohibit the coordination of transportation and inventory decisions. Finally, the development of the capabilities of computers as well as new advances in algorithms allow for the investigation of richer models than in the past.

The scenario we have considered is the following. There is a set of plants and a set of warehouses to deliver the demand to the customers. The production at the plants is constrained and the physical as well as the throughput capacities at the warehouses are restricted. Our goal is to find the most efficient way, i.e. with minimal total costs, to satisfy this demand. We do not allow for transportation between plants. We assume that products are transported to the warehouses immediately, i.e., no storage is allowed at the plants. We do not allow for transportation between warehouses. Customers are supplied by the warehouses under single-sourcing conditions. The customers' demand patterns for a single product are assumed known. The decisions that need to be made are (i) the production sites and quantities, (ii) the assignment of customers to facilities, and (iii) the location and size of inventories.

For a better understanding, we have followed a constructive approach where the optimization model has been enriched gradually. First, we have analyzed the generalized assignment problem (GAP) which can be seen as a static model with only one product and where production and storage take place at the same location. Therefore, we begin to study two-level logistics distribution networks where only one product is handled. Second, we have studied a class of multi-period single-sourcing

problems (MPSSP's) which are dynamic models, integrating transportation and inventory decisions, to evaluate the performance of a logistics distribution network with only one product and where production and storage take place at the same location. In the MPSSP, only the production capacity has been restricted. Additional constraints have been added to the MPSSP. More precisely, we have studied the addition of throughput as well as physical capacities, and perishability constraints. Finally, we have devoted part of this research to the analysis of logistics distribution networks in which the plants and the warehouses are decoupled, i.e., three-level logistics distribution networks.

The main focus of this thesis has been the development of solution procedures for these optimization models. Conclusions about their performance are usually drawn by testing them on a collection of problem instances. For the GAP, we have defined a stochastic model for the problem parameters. We have found conditions on the parameters defining this model which ensure that feasible problem instances are generated with probability one when the number of tasks grows to infinity. Through this stochastic model we have been able to analyze the random generators for the GAP that are used in the literature for empirical testing. From that analysis, we concluded that these random generators are not adequate since they tend to generate easier problem instances, and therefore do not represent the reality, when the number of agents increases. This means that these random generators are not suitable to test the adequacy of solution procedures for the large problem instances appearing in real-life problems. Our stochastic model has overcome this deficiency. Moreover, we can control the tightness of the problem instances generated by this stochastic model.

Based on a reformulation of the GAP and the MPSSP as optimization models in the assignment variables only, we have proposed a class of greedy heuristics and have shown that a particular element from that class yields a greedy heuristic that is asymptotically optimal in a probabilistic sense for the GAP and several large subclasses of the MPSSP. In addition, we have shown that significant improvements can be made by using the result of the greedy heuristic as the starting point of two local exchange procedures, yielding very nearly optimal solutions for problems with many customers.

When production and storage take place at the same location, we have found a set partitioning formulation of these optimization problems, which has been solved by a Branch and Price algorithm. The viability of this approach depends critically on the possibility of solving the pricing problem efficiently. We have identified an important subclass of problems, containing many variants of the MPSSP, for which this is the case. We have applied the method to a particular variant of the MPSSP, and have shown that the Branch and Price algorithm is very useful for solving problems for which the ratio between the number of customers and the number of warehouses is relatively small.

From Chapters 10 and 11, we conclude that the results derived in this thesis may be still suitable for richer and more realistic optimization models to evaluate the design of a logistics distribution network in a dynamic environment. For example, it is possible to deal with several types of products, direct shipments from the plants to the customers, or cross-docking points in the depots.

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Samenvatting

In dit proefschrift hebben we een klasse van optimaliseringsmodellen bestudeerd die bedrijven kunnen gebruiken bij het oplossen van wiskundige problemen die ontstaan als zij inzicht willen krijgen in de mogelijkheden om de efficiëntie van hun logistieke distributienetwerken in een dynamische omgeving te verbeteren. Er waren verschillende redenen om dit onderzoek te starten. Ten eerste is de supply chain, in het Nederlands ook wel aangeduid als de logistieke keten, tegenwoordig aan meer dynamiek onderhevig dan in het verleden. Dit dwingt bedrijven ertoe om regelmatig hun logistieke distributienetwerken te heroverwegen. Ten tweede waren, vanwege de complexiteit van het evalueren van het ontwerp van logistieke distributienetwerken, de meeste optimaliseringsmodellen in de literatuur relatief statisch en was de coördinatie tussen transport- en voorraadbeslissingen omslachtig. Tenslotte laten de ontwikkeling van de reken capaciteit van computers en verbetering van algoritmen het toe om geavanceerdere modellen dan vroeger te onderzoeken.

Het probleem dat we hebben bestudeerd is het volgende. Er zijn een aantal fabrieken met beperkte productiecapaciteit en een aantal depots, die beperkte opslag capaciteit hebben, om de klanten te belevaren. We zoeken de efficiëntste manier, dat wil zeggen minimale totale kosten, om de klanten te belevaren. We nemen aan dat transport tussen de fabrieken onderling niet wordt toegestaan. Bovendien worden producten direct na de productie van de fabrieken naar de depots getransporteerd, oftewel langdurige opslag in de fabrieken is niet toegestaan. Transport tussen de depots onderling is ook niet toegestaan. Klanten worden door de depots belevard onder zogenaamde single-source condities, hetgeen betekent dat iedere klant maar vanuit één depot belevard mag worden. De vraagpatronen van klanten worden bekend verondersteld. De beslissingen die gemaakt moeten worden, hebben betrekking op (i) de productielocaties en -hoeveelheden, (ii) de toewijzing van klanten aan depots en (iii) de voorraadlocaties en -hoeveelheden.

Voor de duidelijkheid hebben we een constructieve benadering gevolgd waarbij het optimaliseringsmodel geleidelijk aan uitgebreid wordt. Als eerste hebben we het zogenaamde Generalized Assignment Problem (GAP) geanalyseerd. Het GAP kan gezien worden als een statisch model met één soort product waarin productie en opslag op dezelfde locaties plaatsvindt en klanten vanuit deze locaties belevard. In feite beschouwen we dus in eerste instantie logistieke distributienetwerken met twee niveaus en slechts één soort product. Vervolgens hebben we om de werking van een logistiek distributienetwerk te evalueren een klasse van zogenaamde meer-

perioden single-sourcing problemen (MPSSP) bestudeerd. Dit zijn dynamische modellen, waarin transport- en opslagbeslissingen geïntegreerd zijn. Ook in deze modellen nemen we aan dat er slechts één soort product is en productie en opslag op dezelfde locaties plaatsvindt. In het MPSSP wordt in eerste instantie alleen de productiecapaciteit begrensd. Later worden additionele restricties aan het MPSSP toegevoegd. In het bijzonder hebben we de toevoeging bestudeerd van restricties op doorvoercapaciteit, opslagcapaciteit en bederfelijkheid. Daarna zijn we de modellen gaan uitbreiden naar logistieke distributienetwerken met drie niveaus, dus modellen waar productie en opslag niet op dezelfde locaties plaatsvindt.

De nadruk ligt in dit proefschrift op de ontwikkeling van oplossingsprocedures voor dit soort optimaliseringsmodellen. Conclusies over de werking van deze procedures worden meestal getrokken door ze te testen op een verzameling van de probleem-instanties. Het genereren van toegelaten probleem-instanties is een probleem op zichzelf. Voor het GAP en het MPSSP hebben we een stochastisch model gedefinieerd voor de generatie van probleemparameters. We hebben voorwaarden voor de parameters afgeleid, die garanderen dat, indien het aantal klanten groeit naar oneindig, met kans 1 toegelaten probleem-instanties gevonden worden. Door dit stochastische model te gebruiken bij de analyse van de bestaande random generatoren voor het GAP, bleek dat de in de literatuur gebruikte random generatoren niet goed zijn omdat ze ertoe neigen om steeds eenvoudigere probleem-instanties te genereren wanneer het aantal depots toeneemt. Hierdoor zijn deze generatoren ongeschikt om te toetsen of oplosmethoden geschikt zijn voor de grotere probleem-instanties zoals die in de praktijk voorkomen. Het door ons voorgestelde stochastische model heeft deze nadelen niet. Bovendien kunnen we controleren of de probleem-instanties die door dit stochastische model gegenereerd worden veel ruimte voor toegelaten oplossingen bieden.

Op basis van een herformulering van het GAP en het MPSSP, als optimaliseringsmodellen met alleen toewijzingsvariabelen, hebben we een klasse van zogenaamde gretige heuristieken ontwikkeld. Bovendien laten we zien dat een element uit deze klasse asymptotisch toegelaten en optimaal is, in de probabilistische zin, voor het GAP en voor verschillende grote sub-klassen van MPSSP's. Daarnaast hebben we laten zien dat significante verbeteringen geboekt kunnen worden door het resultaat van de gretige heuristiek te gebruiken als startpunt van twee zogenaamde lokale uitwisselingsprocedures. Dit levert bijna optimale oplossingen op voor problemen met veel klanten.

Wanneer productie en opslag op dezelfde locaties plaatsvindt, hebben we een set partitie formulering gevonden van deze optimaliseringsmodellen die opgelost kan worden met behulp van een Branch-and-Price algoritme. De bruikbaarheid van deze benadering hangt in grote mate af van de mogelijkheid om binnen dit algoritme het zogenaamde pricing probleem efficiënt op te lossen. We hebben een belangrijke subklasse van problemen, die het GAP en veel varianten van het MPSSP bevat, gevonden waarvoor dit het geval is. Daarnaast hebben we de methode toegepast op een bijzondere variant van het MPSSP en hebben laten zien dat het Branch-and-Price algoritme erg nuttig is om problemen op te lossen waarbij de ratio tussen het aantal klanten en het aantal depots relatief klein is.

Op basis van de resultaten in Deel IV van dit proefschrift mogen we zeggen dat de bereikte resultaten geschikt zijn om complexere meer praktische logistieke distributie problemen in een dynamische omgeving te evalueren. Zo is het bijvoorbeeld mogelijk om de modellen uit te breiden naar meerdere soorten producten, directe levering van de fabrieken naar de klanten, of problem met zogenaamde cross-docking in depots.

Curriculum Vitae

María Dolores Romero Morales was born on Augustus 5th, 1971, in Sevilla (Spain). She studied Mathematics at University of Sevilla from 1989 to 1994 and specialized in Statistics and Operations Research. She wrote her Master's thesis on Global Optimization in Location Theory under the supervision of Dr. Emilio Carrizosa Priego and Dr. Eduardo Conde Sánchez. During the academic year 1995-1996 she was assistant professor at the department of Mathematics of the Business School, University of Cádiz, Spain. From September 1996 to July 2000, she has been research assistant at the department of Decision and Information Sciences of the Faculty of Business Administration, Erasmus University, The Netherlands. During the second semester of the academic year 1997-1998, she was assistant professor at the department of Statistics and Operations Research of the Faculty of Mathematics, University of Sevilla, Spain. Currently, she is an assistant professor at the Faculty of Economics and Business Administration, Maastricht University, The Netherlands.